$$\text{equation1} = x\frac{\partial u[x,y]}{\partial x} + y\frac{\partial u[x,y]}{\partial y} = n\,u[x,y];$$

```
sol = u /. DSolve[equation1, u, {x, y}][[1]] /. C[1][t_] → t
```

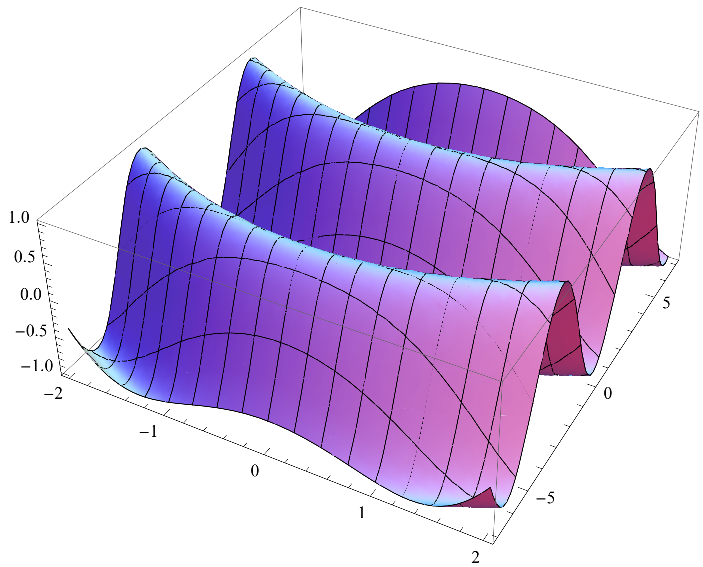
$$\text{Function}\left[\{x, y\}, \frac{x^n\, y}{x}\right]$$

The Cauchy ' s Equations :

```
eqn = ∂x u[x, y] + x ∂y u[x, y] == 0;
sol = u[x, y] /. DSolve[{eqn, u[0, y] == Sin[y]}, u[x, y], {x, y}]
```

$$\left\{\text{Sin}\left[\frac{1}{2}\left(-x^2 + 2\,y\right)\right]\right\}$$

```
Plot3D[sol, {x, -2, 2}, {y, -7, 7}, PlotPoints → 30]
```



Example . Find the charecteristic of the equation $(u - y)\, u_x + y u_y = x + y$

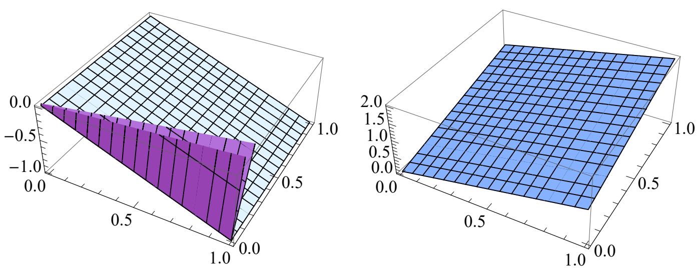
Solution. The characteristic system is

$\frac{dx}{u - y} = \frac{dy}{y} = \frac{du}{x + y}$. **Using (i) + (iii) = (ii),**

we have then $v = \frac{u + x}{y} = c1$, **is a first integral. From (i) + (ii) = (iii),**

it follows that w = (x + y) ^2 - u^2 = c^2 is a second first integral.

```
f0 = Plot3D[-x, {x, 0, 1}, {y, 0, 1}, PlotPoints → 10];
f1 = Plot3D[5 y - x, {x, 0, 1}, {y, 0, 1}, PlotPoints → 10];
 f2 = Plot3D[10 y - x, {x, 0, 1}, {y, 0, 1}, PlotPoints → 10];
g1 = Show[f0, f1, f2];
h0 = Plot3D[x + y, {x, 0, 1}, {y, 0, 1}, PlotPoints → 10];
 h1 = Plot3D[Sqrt[(x + y) ^2 + 5], {x, 0, 1}, {y, 0, 1}, PlotPoints → 10];
 h2 = Plot3D[Sqrt[(x + y) ^2 + 10], {x, 0, 1}, {y, 0, 1}, PlotPoints -> 10];
g2 = Show[h0, h1, h2];
Show[GraphicsGrid[{{g1, g2}}]]
```



Integral Surface Plots :

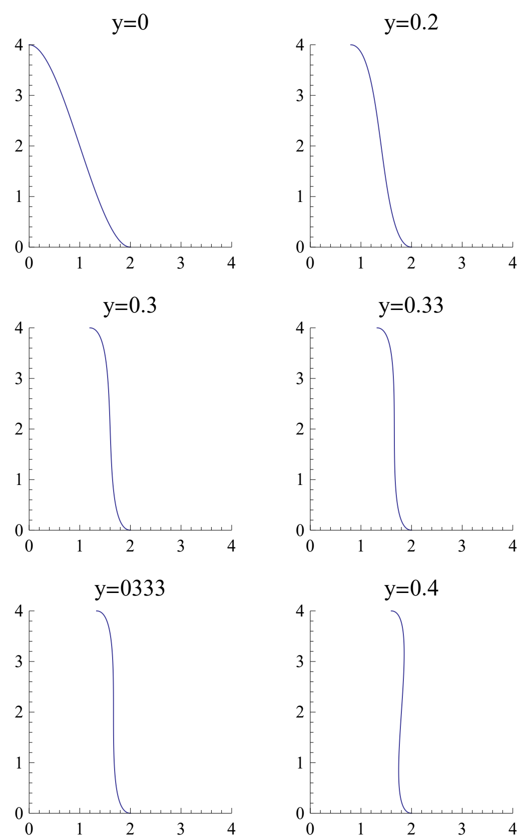
Example : Solve the PDE $uu_x + u_y = 1/2$, with initial condition u (s, s) = s / 4, 0 ≤ s ≤ 1.
Solution : x = s + st / 4 + t^2 / 4, y = s + t, u = s / 4 + t / 2.

```
f1 = ParametricPlot3D[{s + ((t^2 + s t) / 4), t + s, (2 t + s) / 4},
   {s, 0, 1}, {t, -1, 1}, PlotPoints → 10]

f2 = ParametricPlot3D[{s, s, s / 4}, {s, -0.5, 1.5}]

Show[f1, f2, PlotLabel → "Integral  surface  through  initial  curve"]
```

Example 2. The solution of the equation $u_y + uu_x = 0$,
can be interpreted as a vector field on the x - axis varying with the time y. Find the integral surface satisfying the initial condition u (s, 0) = h (s), where h is a given function.
Solution : consider uo (s) = s^3 - 3 s^2 + 4,
0 ≤ s ≤ 2. We plot the curves { Ct : x = s + t (s^3 - 3 s^2 + 4), u = s^3 - 3 s^2 + 4}

```
u[s_] := s^3 - 3 s^2 + 4;

x[s_, t_] := s + t u[s];
```

```
h0 = ParametricPlot[{x[s, 0], u[s]}, {s, 0, 2}, PlotRange → {0, 4}, PlotLabel → "y=0"];
h1 = ParametricPlot[{x[s, 0.2], u[s]},
   {s, 0, 2}, PlotRange → {0, 4}, PlotLabel → "y=0.2"];
h2 = ParametricPlot[{x[s, 0.3], u[s]}, {s, 0, 2},
   PlotRange → {0, 4}, PlotLabel → "y=0.3"];
h3 = ParametricPlot[{x[s, 0.33], u[s]}, {s, 0, 2},
   PlotRange → {0, 4}, PlotLabel → "y=0.33"];
h4 = ParametricPlot[{x[s, 0.333], u[s]}, {s, 0, 2},
   PlotRange → {0, 4}, PlotLabel → "y=0333"];
h5 = ParametricPlot[{x[s, 0.4], u[s]}, {s, 0, 2},
   PlotRange → {0, 4}, PlotLabel → "y=0.4"];
Show[GraphicsArray[{{h0, h1}, {h2, h3}, {h4, h5}}], FrameTicks -> None, Frame → False]
```



The Wave Equation :

(1) $u_{tt} - c^2 u_{xx} = F(x, t)$, $0 < x < a$, $t > 0$, $u(x, 0) = f(x)$,
$u_t(x, 0) = g(x)$ $c = 1$; $F = 0$, $f = \text{Sin}\,x$; $g = x^2$;

```
wws = {∂t,t u[x, t] - ∂x,x u[x, t] == 0, u[x, 0] == Sin[x], Derivative[0, 1][u][x, 0] == x^2};
```

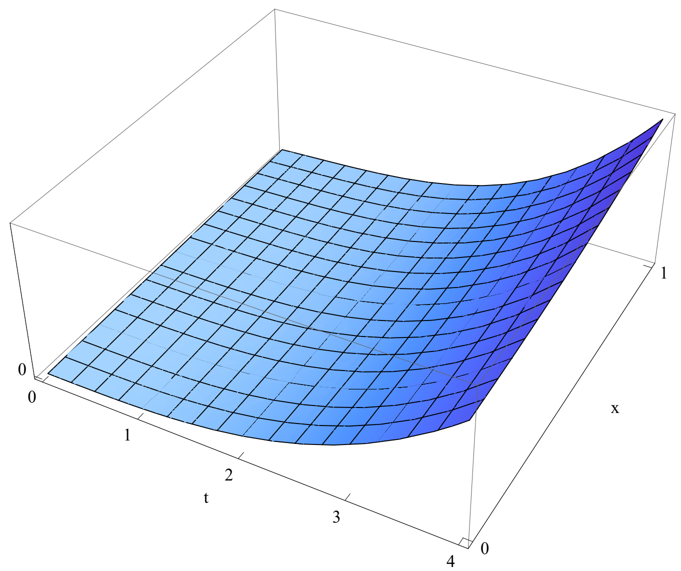
```
sol2 = u[x, t] /. NDSolve[wws, u[x, t], {x, 0, 1}, {t, 0, 4}, PrecisionGoal → 3][[1]]
```

NDSolve::bcart :

Warning: An insufficient number of boundary conditions have been specified for the direction of independent variable x. Artificial boundary effects may be present in the solution. ≫

```
InterpolatingFunction[{{0., 1.}, {0., 4.}}, <>][x, t]
```

```
Plot3D[sol2, {t, 0, 4}, {x, 0, 1},
 AxesLabel → {"t", "x", ""}, Ticks → {{0, 1, 2, 3, 4}, {0, 1}, {-3, 0}}]
```



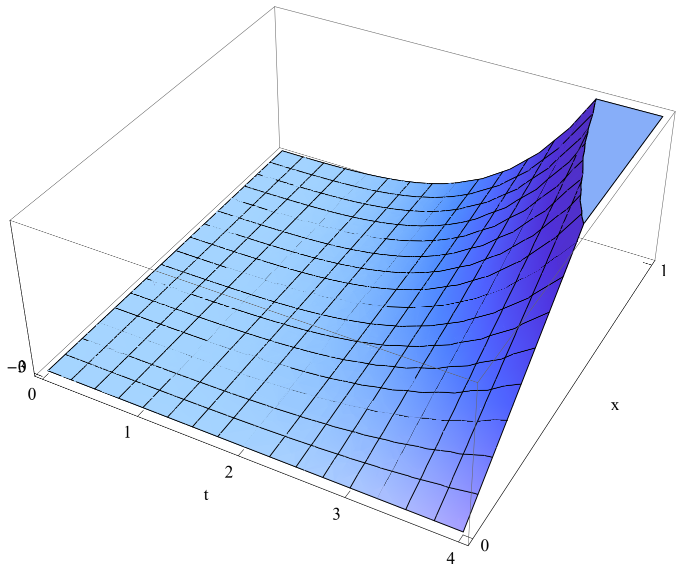
(2) $u_{tt} - c^2 u_{xx} = F(x, t)$, $0 < x < a$, $t > 0$, $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$, $c = 2$; $F = 0$, $f = x^4$; $g = 0$, $u(0, t) = 0$;

```
eqns = {∂t,t u[x, t] - 4 ∂x,x u[x, t] == 0,
   u[x, 0] == 10 x^4, Derivative[0, 1][u][x, 0] == 0, u[0, t] == 0};
```

```
sol1 = u[x, t] /. NDSolve[eqns, u[x, t], {x, 0, 1}, {t, 0, 4}, PrecisionGoal → 3][[1]]
```

```
InterpolatingFunction[{{0., 1.}, {0., 4.}}, <>][x, t]
```

```
Plot3D[sol1, {t, 0, 4}, {x, 0, 1},
 AxesLabel → {"t", "x", ""}, Ticks → {{0, 1, 2, 3, 4}, {0, 1}, {-3, 0}}]
```



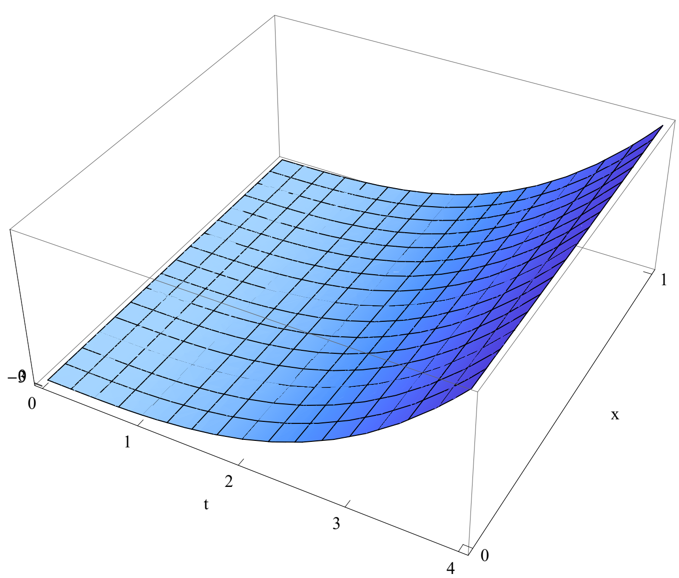
(3) $u_{tt} - c^2 u_{xx} = F(x, t)$, $0 < x < a$, $t > 0$, $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$, $c = 3$; $F = 0$, $f = 0$; $g = x^3$, $u_x(0, t) = 0$;

```
eqns1 = {∂t,t u[x, t] - 9 ∂x,x u[x, t] == 0, u[x, 0] == 0,
   Derivative[0, 1][u][x, 0] == x^3, Derivative[1, 0][u][0, t] == 0};
```

```
sol1 = u[x, t] /. NDSolve[eqns1, u[x, t], {x, 0, 1}, {t, 0, 4}, PrecisionGoal → 3][[1]]
```

```
InterpolatingFunction[{{0., 1.}, {0., 4.}}, <>][x, t]
```

```
Plot3D[sol1, {t, 0, 4}, {x, 0, 1},
 AxesLabel → {"t", "x", ""}, Ticks → {{0, 1, 2, 3, 4}, {0, 1}, {-3, 0}}]
```



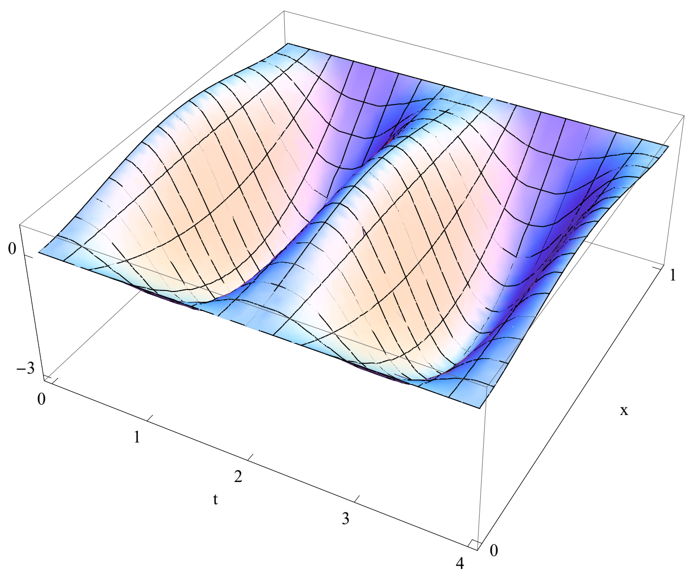
(4) $u_{tt} - c^2 u_{xx} = F(x, t)$, $0 < x < a$, $t > 0$,
$u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$, $u(0, t) = u(a, t) = 0$.
$a = 1$; $c = 1$; $F = -9.80665$, $f = 10 x^2 (1 - x)^2$; $g = 0$;

```
eqns = {∂t,t u[x, t] - ∂x,x u[x, t] == -9.80665, u[x, 0] == 10 x^2 (1 - x)^2,
   Derivative[0, 1][u][x, 0] == 0, u[0, t] == 0, u[1, t] == 0};
```

```
sol1 = u[x, t] /. NDSolve[eqns, u[x, t], {x, 0, 1}, {t, 0, 4}, PrecisionGoal → 3][[1]]
```

```
InterpolatingFunction[{{0., 1.}, {0., 4.}}, <>][x, t]
```

```
Plot3D[sol1, {t, 0, 4}, {x, 0, 1},
 AxesLabel → {"t", "x", ""}, Ticks → {{0, 1, 2, 3, 4}, {0, 1}, {-3, 0}}]
```



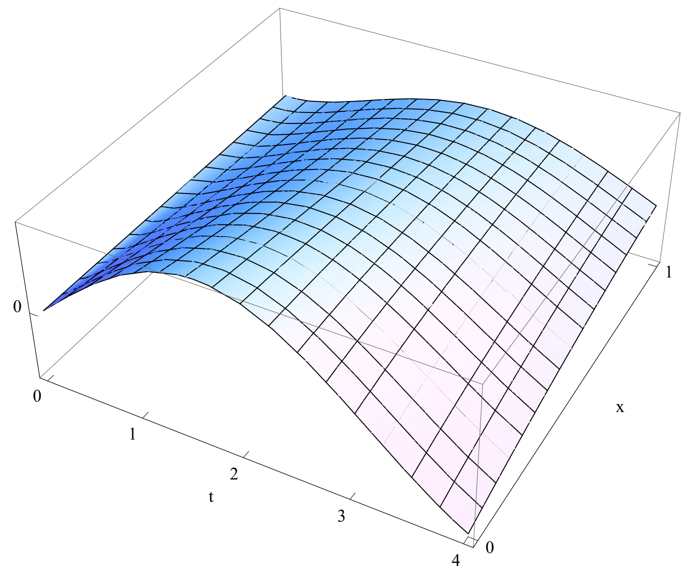
The Heat Equation :

```
he1 = {∂t u[x, t] - ∂x,x u[x, t] == 0, u[0, t] == Sin[t], u[x, 0] == 0, u[5, t] == 0};
```

```
sol1 = u[x, t] /. NDSolve[he1, u[x, t], {x, 0, 5}, {t, 0, 10}, PrecisionGoal → 3][[1]]
```

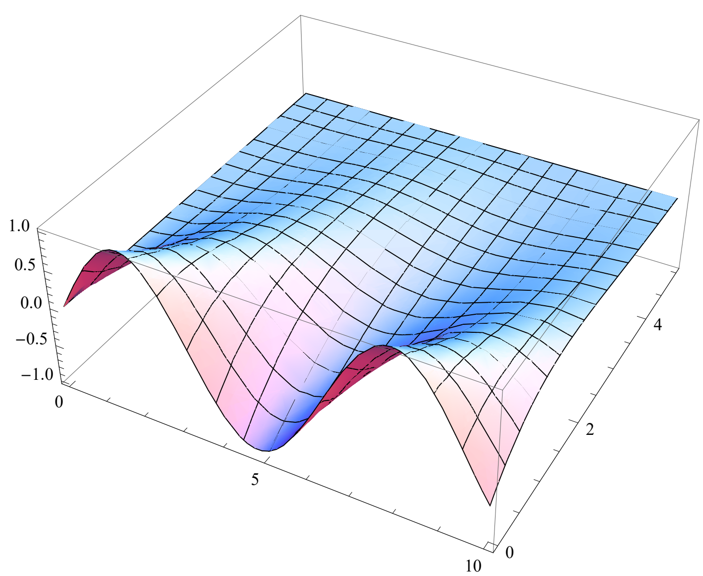
```
InterpolatingFunction[{{0., 5.}, {0., 10.}}, <>][x, t]
```

```
Plot3D[sol1, {t, 0, 4}, {x, 0, 1},
 AxesLabel → {"t", "x", ""}, Ticks → {{0, 1, 2, 3, 4}, {0, 1}, {-3, 0}}]
```




```
NDSolve[{D[u[t, x], t] == D[u[t, x], x, x], u[0, x] == 0,
  u[t, 0] == Sin[t], u[t, 5] == 0}, u, {t, 0, 10}, {x, 0, 5}]
{{u → InterpolatingFunction[{{0., 10.}, {0., 5.}}, <>]}}
```

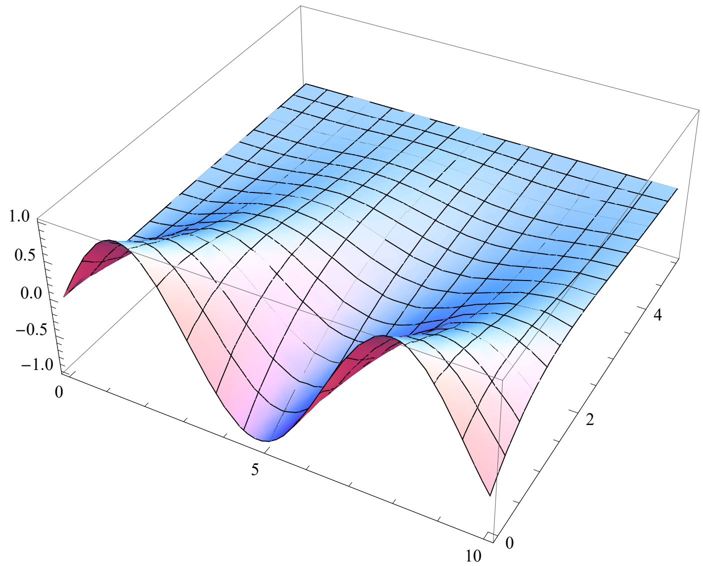
```
Plot3D[Evaluate[u[t, x] /. %], {t, 0, 10}, {x, 0, 5}, PlotRange → All]
```



Alternative form of equation:

```
NDSolve[{∂t u[t, x] == ∂x,x u[t, x], u[0, x] == 0, u[t, 0] == Sin[t], u[t, 5] == 0},
 u, {t, 0, 10}, {x, 0, 5}]
{{u → InterpolatingFunction[{{0., 10.}, {0., 5.}}, <>]}}
```

```
Plot3D[Evaluate[u[t, x] /. %], {t, 0, 10}, {x, 0, 5}, PlotRange → All]
```




```
Euler[x_, w_] := w + h (4 x^2 - 2 w)
h = 0.1; w = 2
w = Euler[0, w]
w = Euler[0.1, w]
w = Euler[0.2, w]
w = Euler[0.3, w]
w = Euler[0.4, w]
w = Euler[0.5, w]
w = Euler[0.6, w]
w = Euler[0.7, w]
w = Euler[0.8, w]
w = Euler[0.9, w]
```

```
2

1.6

1.284

1.0432

0.87056

0.760448

0.708358

0.710687

0.764549

0.86764

1.01811
```

Practical 9

Draw the following sequences of functions on the given interval, and discuss the uniform convergence.

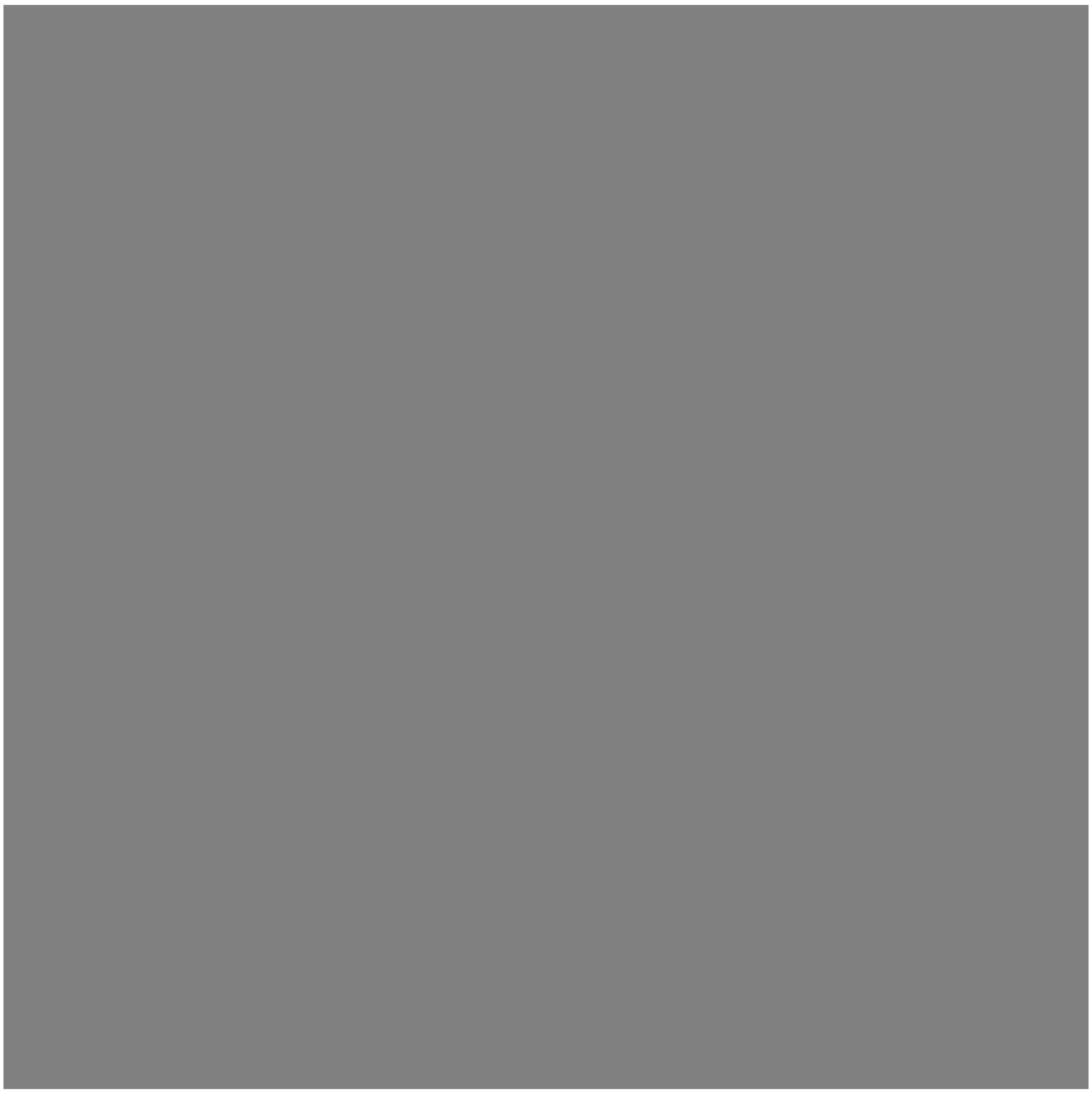
(1) $f_n(x) = x^n, \ x \in \mathbb{R}$

```
Manipulate[Plot[Table[x^n, {n, m}], {x, -2, 2},
  PlotRange → {-2, 2}, PlotStyle → {Orange, Thick},
  Epilog → {Opacity[.1], LightOrange, EdgeForm[
     GrayLevel[.1]], Rectangle[{-a, l - ϵ}, {a, l + ϵ}]}],
 {m, 1, 40, 1, Appearance → "Labeled"},
 {ϵ, 0.01, 0.5, 0.001, Appearance → "Labeled"},
 {a, 0, 2, 0.01, Appearance → "Labeled"},
 {l, 0, 2, 0.01, Appearance → "Labeled"}]
```

Conclusion : The given sequence of functions converges uniformly to the function f(x) = 0 in any interval [-k,k], 0<k<1.

(2) $f_n(x) = \frac{x}{n}, \; x \in \mathbb{R}$

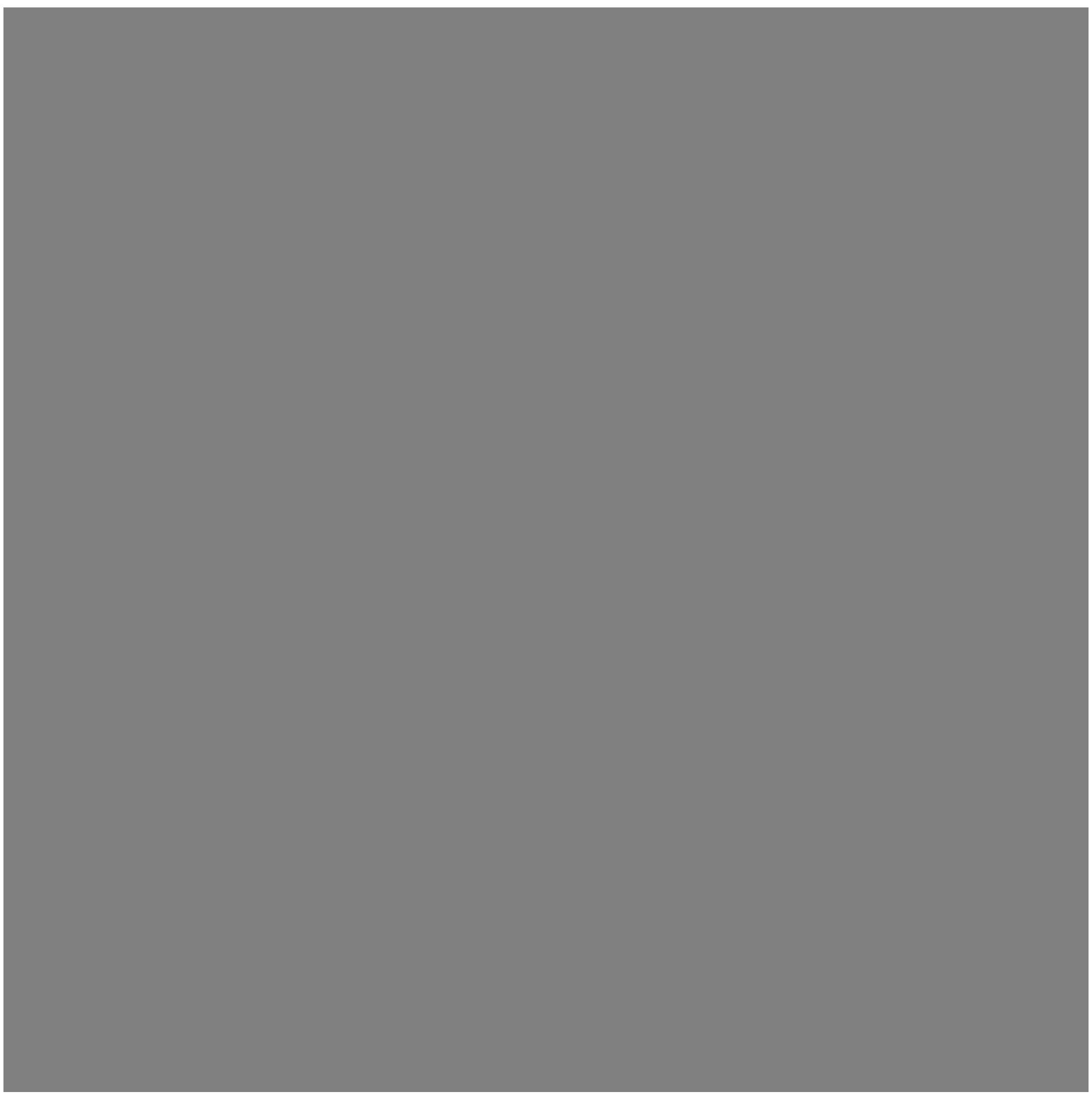
```
Manipulate[Plot[Table[x/n, {n, m}], {x, -5, 5}, PlotRange → {-2, 2},
  PlotStyle → {Magenta, Thick}, Epilog → {Opacity[.5], LightOrange,
   EdgeForm[GrayLevel[.6]], Rectangle[{-a, l - ϵ}, {a, l + ϵ}]}],
 {ϵ, 0.01, 0.5, 0.001, Appearance → "Labeled"},
 {a, 0, 5, 0.01, Appearance → "Labeled"},
 {m, 1, 40, 1, Appearance → "Labeled"},
 {l, 0, 2, 0.01, Appearance → "Labeled"}]
```



Conclusion : The given sequence of functions converges uniformly to the function f(x) = 0 in any interval [-k, k], k>0.

(3) $f_n(x) = \frac{x^2+n\,x}{n}, \; x \in \mathbb{R}$

```
Manipulate[Plot[Table[(x^2 + n x)/n, {n, m}], {x, -5, 5},
  PlotRange → {-4, 4}, PlotStyle → {Pink, Thick},
  Epilog → {Opacity[.5], LightOrange,
    EdgeForm[GrayLevel[.7]], Rectangle[{0, l - ϵ}, {a, l + ϵ}]}],
 {ϵ, 0.01, 0.5, 0.001, Appearance → "Labeled"},
 {a, 0, 2, 0.01, Appearance → "Labeled"},
 {m, 1, 50, 1, Appearance → "Labeled"},
 {l, 0, 2, 0.01, Appearance → "Labeled"}]
```



Conclusion : The given sequence of functions converges uniformly to the function f(x) = x in any interval [-k, k], k > 0.

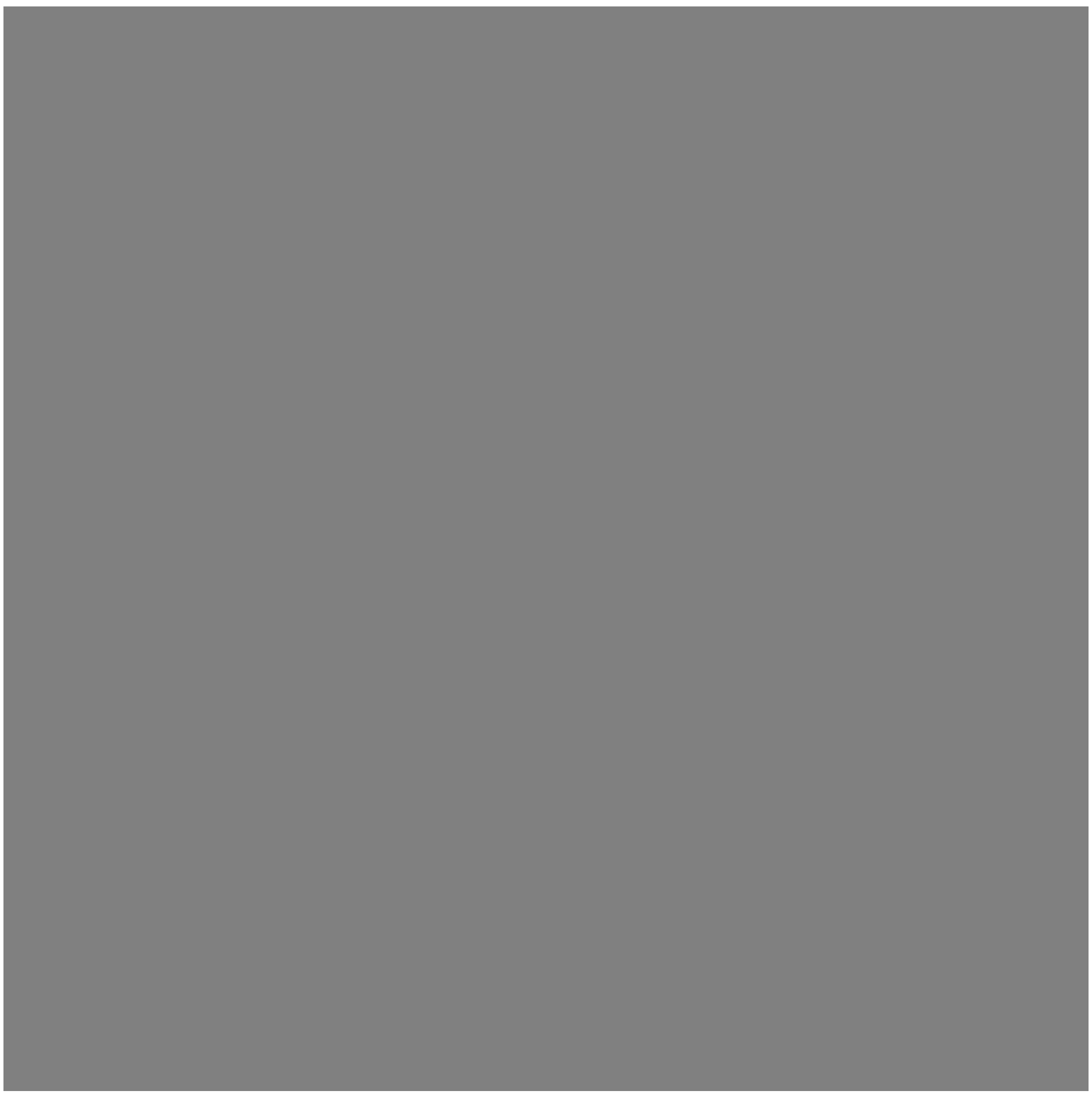
(4) $f_n(x) = \dfrac{x}{x+n}$, $x \in \mathbb{R}$, $x \geq 0$

```
Manipulate[Plot[Table[x/(x + n), {n, m}],
   {x, 0, 20}, PlotRange → {-0.5, 1.5}, PlotStyle → Thick,
   Epilog → {Opacity[.5], LightOrange,
     EdgeForm[GrayLevel[.7]], Rectangle[{0, l - ϵ}, {a, l + ϵ}]}],
 {ϵ, 0.01, 0.5, 0.001, Appearance → "Labeled"},
 {a, 0, 20, 0.01, Appearance → "Labeled"},
 {m, 1, 50, 1, Appearance → "Labeled"},
 {l, 0, 2, 0.01, Appearance → "Labeled"}]
```

Conclusion : The given sequence of functions converges uniformly to 0 in any interval [-k, k], k > 0.

(5) $f_n(x) = \frac{n\,x}{1 + n^2\,x^2}$, $x \in \mathbb{R}$

```
Manipulate[Plot[Table[(n x)/(1 + n^2 x^2), {n, m}],
  {x, -2, 2}, PlotRange → {-1, 1}, PlotStyle → Red,
  Epilog → {Opacity[.5], LightOrange,
    EdgeForm[GrayLevel[.7]], Rectangle[{1, l - ϵ}, {a, l + ϵ}]}],
 {ϵ, 0.01, 0.5, 0.001, Appearance → "Labeled"},
 {a, 1, 2, 0.01, Appearance → "Labeled"},
 {m, 1, 30, 1, Appearance → "Labeled"},
 {l, 0, 2, 0.01, Appearance → "Labeled"}]
```



Conclusion : The given sequence of functions converges uniformly to 0 in any interval [-k, k], k > 0.

(6) $f_n(x) = \dfrac{n x}{1 + n x}, \ x \in \mathbb{R}, \ x \geq 0$

```
Manipulate[Plot[Table[(n x)/(1 + n x), {n, m}], {x, 0, 1},
  PlotRange → {-1.5, 1.5}, PlotStyle -> {Brown, Thick},
  Epilog → {Opacity[.5], LightOrange,
    EdgeForm[GrayLevel[.7]], Rectangle[{0, l - ϵ}, {a, l + ϵ}]}],
 {ϵ, 0.01, 0.5, 0.001, Appearance → "Labeled"},
 {a, 0, 1, 0.01, Appearance → "Labeled"},
 {m, 1, 50, 1, Appearance → "Labeled"},
 {l, 0, 2, 0.01, Appearance → "Labeled"}]
```

Conclusion : The given sequence of functions converges uniformly to 0 in any interval [-k, k], k > 0.

(7) $f_n(x) = \dfrac{x^n}{1 + x^n}, \; x \in \mathbb{R}, \; x \geq 0$

```
Manipulate[ Plot[Table[x^n/(1 + x^n), {n, m}], {x, 0, 5},
   PlotRange → {-0.5, 1.5}, PlotStyle → {Magenta, Thick},
   Epilog → {Opacity[.5], LightOrange,
     EdgeForm[GrayLevel[.7]], Rectangle[{0, l - ϵ}, {a, l + ϵ}]}],
  {ϵ, 0.01, 0.5, 0.001, Appearance → "Labeled"},
  {a, 0, 5, 0.01, Appearance → "Labeled"},
  {m, 1, 40, 1, Appearance → "Labeled"},
  {l, 0, 1.5, 0.01, Appearance → "Labeled"}]
```

Conclusion : The given sequence of functions converges uniformly to 0 in any interval [-k, k], k > 0.