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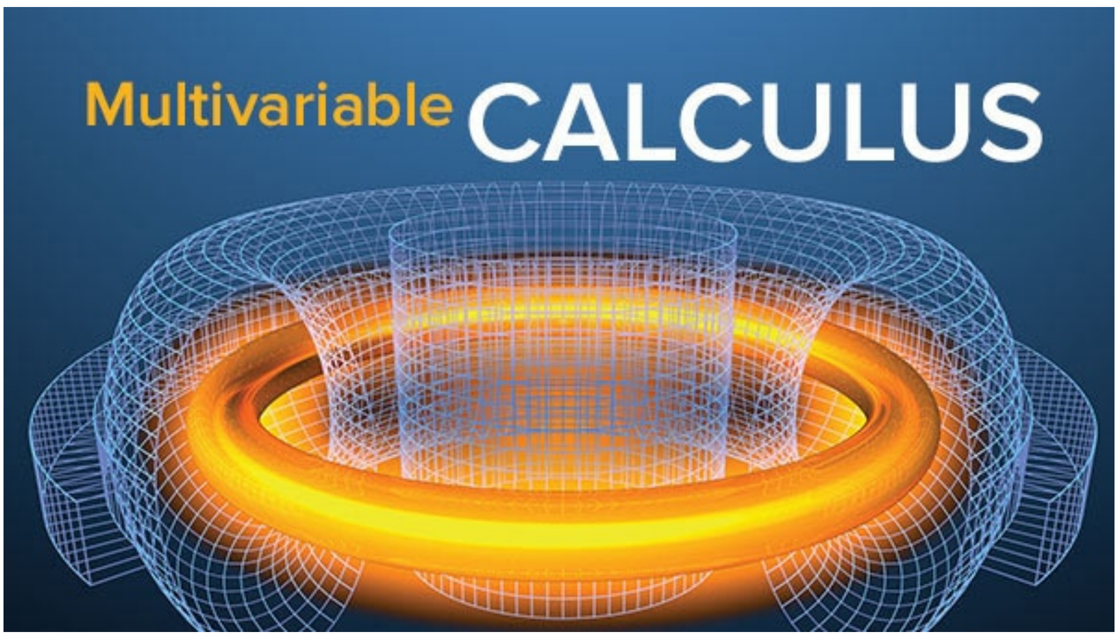
MAT / 19 / 85

MULTIVARIATE CALCULUS

What is Multivariate Calculus?

Multivariate Calculus is the extension of calculus in one variable to calculus with functions of several variables that is the differentiation and integration of functions involving several variables, rather than just one.



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Plotting functions of in 3D

Plotting functions of 2 variables

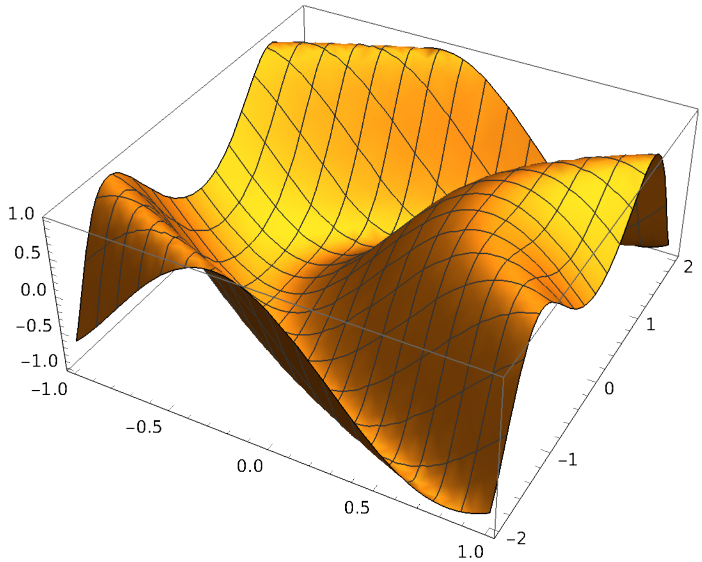
1) f (x) = Sin[3 x - y ^ 2]

In[5]:=
```
f[x_, y_] := Sin[3 x - y ^ 2]
```

In[6]:=

```
Plot3D[f[x, y], {x, -1, 1}, {y, -2, 2}]
```

Out[6]=

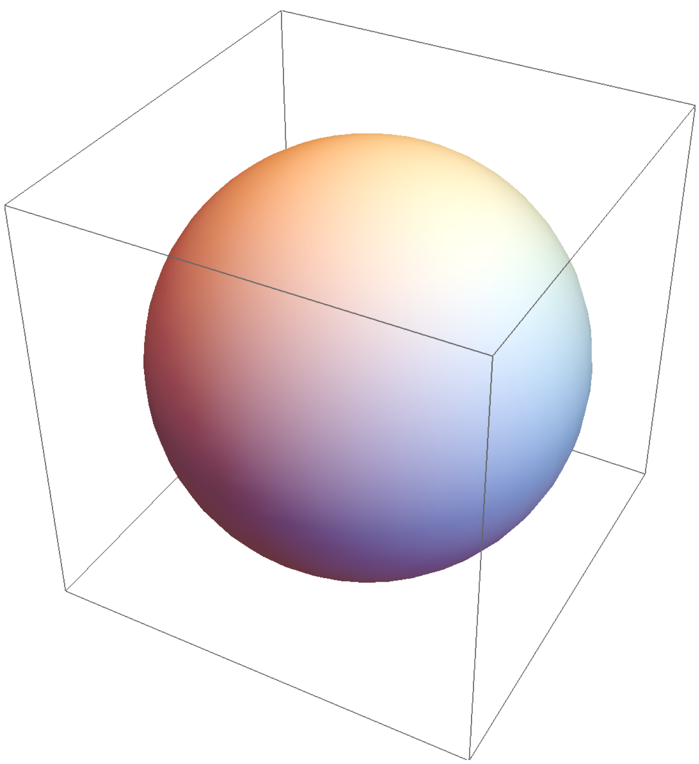


Plotting functions of 3 variables

In[9]:=

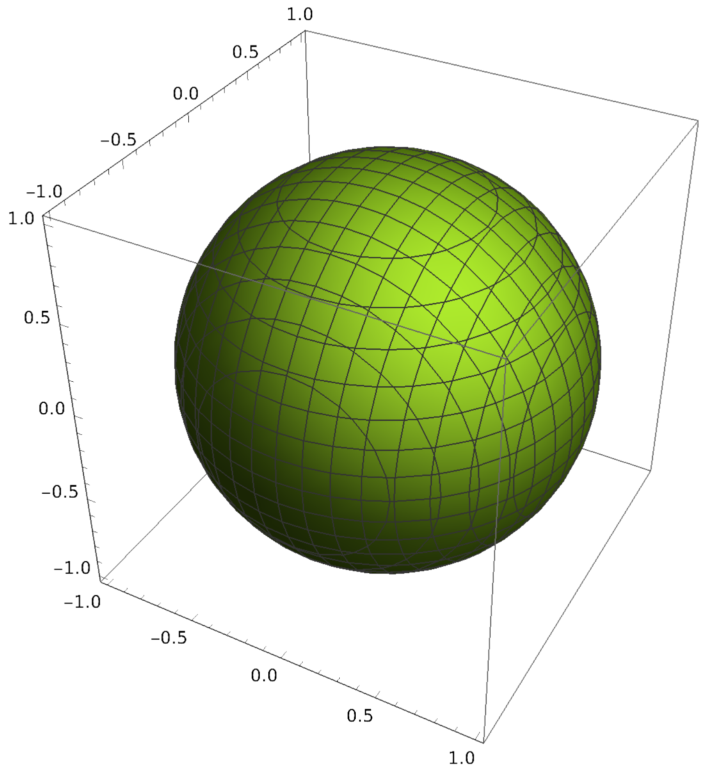
```
Graphics3D[Sphere[{0, 0, 0}, 3]]
```

Out[9]=



In[11]:=
```
ContourPlot3D[x^2 + y^2 + z^2 == 1, {x, -1, 1}, {y, -1, 1}, {z, -1, 1}]
```

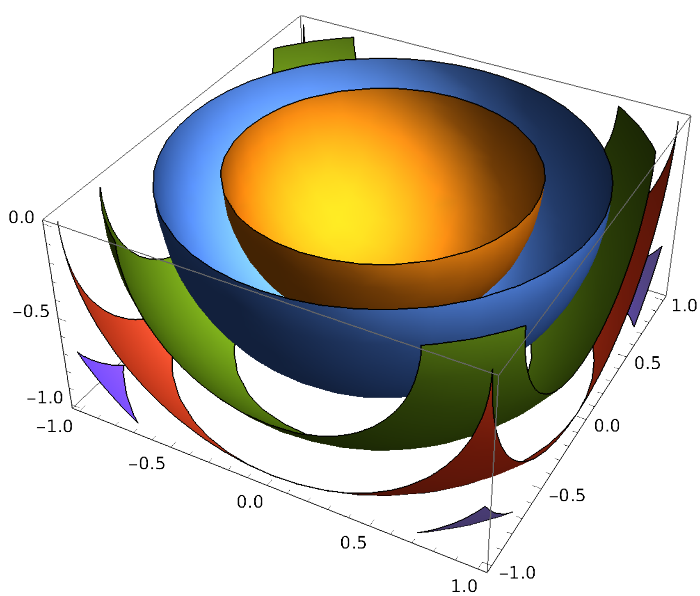
Out[11]=



plotting of level surfaces.

In[14]:=
```
ContourPlot3D[x^2 + y^2 + z^2, {x, -1, 1}, {y, -1, 1},
 {z, -1, 0}, BoxRatios → {2, 2, 1}, Contours → 5, Mesh → None]
```

Out[14]=



LIMIT AND CONTINUITY

defn : The function f (x, y) = z is **CONTINUOUS** at point (x1, y1) if →

▪f (x1, y1) is defined

▪lim (x, y) → (x1, y1) of f (x, y) exists.

▪ lim (x, y) → (x1, y1) of f (x, y) = f (x1, y1)

NOTE :

1) Composition of 2 continuous functions is also continuous.

2) Every polynomial is continuous.

3) Rational funtions are
continuous provided the denominator is non – 0

TO CHECK WHETHER THE LIMIT EXISTS AND COMMENT ON THE CONTINUITY.

h[x,y]= (x+y)/(x-y)^(Exp[x*y])

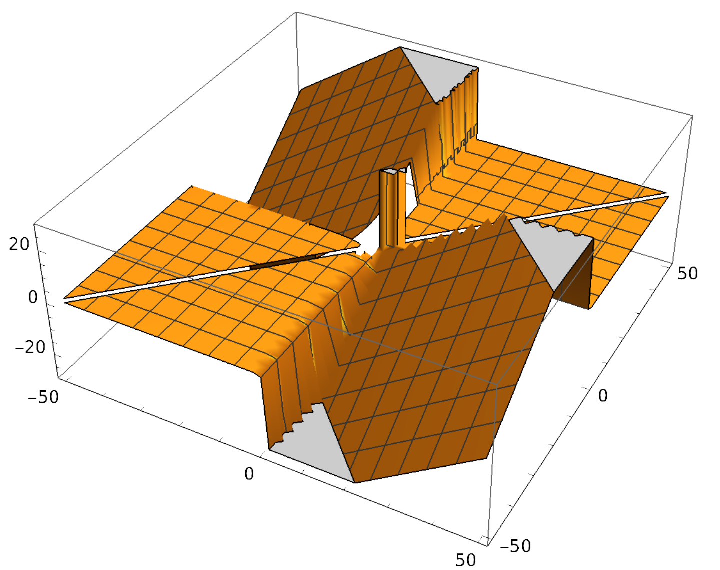
```
In[49]:= h[x_, y_] := (x + y) / (x - y) ^ (Exp[x * y])

In[50]:= Plot3D[h[x, y], {x, -50, 50}, {y, -50, 50}]
```

Power : Infinite expression $\frac{1}{0.}$ encountered .

General : Overflow occurred in computation .

Out[50]=




```
In[3]:= Limit[h[x, y], {x, y} → {1, 0}]

Out[3]= 1
```

In[4]:=
```
Limit[h[x, y], {x, y} → {1, 1}]
```
Out[4]= Indeterminate

along X axis

In[6]:=
```
Limit[h[x, 0], {x → 1}]
```
Out[6]= 1

along Y axis

In[7]:=
```
Limit[h[0, y], {y → 1}]
```
Out[7]= -1

Since the limits are not equal of the above paths, the limit does not exist .
And is not continuous.

TANGENT PLANES

Defn : Well tangent planes to a surface are planes that just touch the surface at the point and are “parallel” to the surface at the point. Note that this gives us a point that is on the plane.
equation of the tangent planeat point (x0, y0, z0) → z =
z0 + f_x (x - x0) + f_y (y - y0)

Find the equation of tangent planes at the given point.

1) g(x,y,z)= x^2+y^2+z^2-9 at (3,0,0) where g is a function of 3 variables.

In[1]:=
```
g[x_, y_, z_] := x^2 + y^2 + z^2 - 9
```
In[29]:=
```
a = 3;
```
In[30]:=
```
b = 0;
```
In[31]:=
```
c = 0;
```
In[34]:=
```
A = D[g[x, y, z], x] /. {x → a, y → b, z → c}
```
Out[34]= 6

In[35]:=
```
B = D[g[x, y, z], y] /. {x → a, y → b, z → c}
```
Out[35]= 0

In[37]:=
```
M = D[g[x, y, z], z] /. {x → a, y → b, z → c}
```
Out[37]= 0

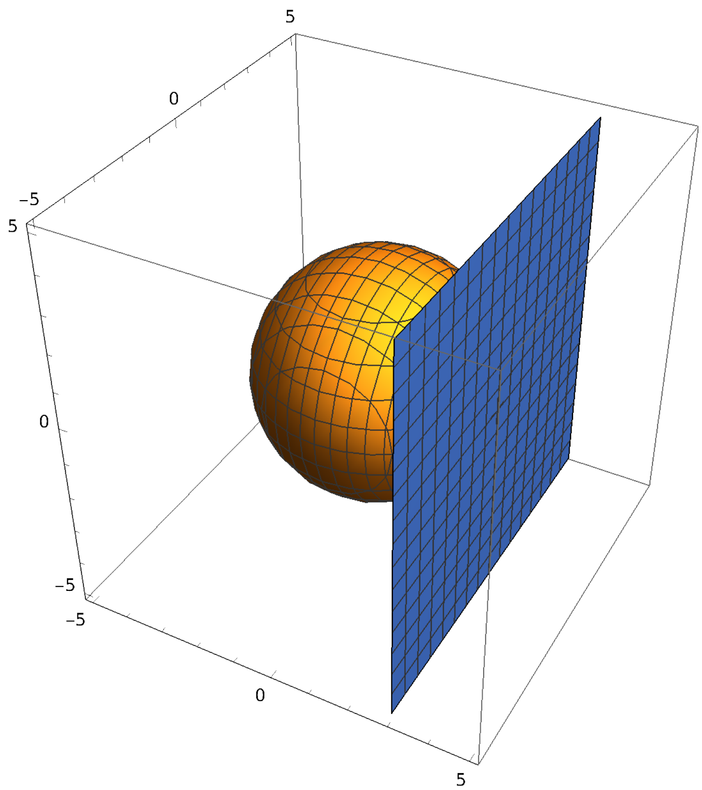
```
In[40]:= w =  c + A (x - a) + B (y - b) + M (z - c) == 0
Out[40]= 6 (-3 + x) == 0

In[43]:= Simplify[w]
Out[43]= x == 3

In[44]:= Print["The tangent plane equation is given by ", %]
         The tangent plane equation is given by x == 3

In[2]:= ContourPlot3D[{g[x, y, z] == 0, x == 3}, {x, -5, 5}, {y, -5, 5}, {z, -5, 5}]
```

Out[2]=



INCREMENTAL APPROXIMATION.

defn : THe increment represents the change in the value of f when (x, y) changes from (x0, y0) to (x0 + Δx, y0 + Δy)

IF |Δx| and |Δy| are very small then →

f[x0 + Δx , y0 + Δy] ≃ f_x (x0, y0) Δx + f_y (x0, y0) Δy + f (x0, y0)

and change in f is given by →

Δf = f (x0 + Δx, y0 + Δy) - f (x0, y0)

USE INCREMENTAL APPROXIMATION TO ESTIMATE THE FOLLOWING FUNCTION AT GIVEN POINT AND COMPARE IT WITH THE CALCULATED VALUE

1) f(x,y)= e^xy at (1.01,0.98)

```
In[60]:= f[x_, y_] := Exp[x * y]

In[61]:= a = 1;

In[62]:= b = 1;

In[63]:= Δa = 1.01-a
         Δb = 0.98-b

Out[63]= 0.01

Out[64]= -0.02

In[65]:= A = D[f[x, y], x] /. {x → a, y → b}

Out[65]= e

In[66]:= B = D[f[x, y], y] /. {x → a, y → b}

Out[66]= e

In[67]:= z = A * Δa + B Δb + f[a, b]

Out[67]= 2.6911

In[68]:= Print["Approximate  value of f at [1.01,0.98] is ", %]

         Approximate  value  of  f  at [1.01 ,0.98] is 2.6911

In[69]:= Print["Actual value of f at [1.01,0.98] is ", f[a + Δa , b + Δb]]

         Actual  value  of  f  at [1.01 ,0.98] is 2.6907

In[70]:= difference = 2.6907 - 2.6911

Out[70]= -0.0004

In[71]:= Print["percentage  change /error  is ", Abs[% / 2.6907] * 100]

         percentage  change  /error  is 0.014866

In[73]:= ClearAll[f, a, b, c, A, B, z, Δa, Δb]
```

CRITICAL POINTS, RELATIVE MAXIMA AND MINIMA

A **critical point** of a multivariable function is a point where the partial derivatives of first order of this function are equal to zero.

SECOND PARTIAL DERIVATIVE TEST.

Suppose that f (x, y) is a differentiable real function of two variables whose second partial derivatives exist and are continuous .Define D (x, y) to be the determinant
D (x, y) = f_xx (x, y) * f_yy (x, y) - (f_xy (x, y))^2
Finally , suppose that (a, b) is a critical point of f
(that is, fx (a, b) = fy (a, b) = 0). Then the second partial derivative test asserts the following :

- If D (a, b) > 0 and fxx (a, b) > 0 then (a, b) is a relative minimum of f.
- If D (a, b) > 0 and fxx (a, b) < 0 then (a, b) is a relative maximum of f.
- If D (a, b) < 0 then (a, b) is a saddle point of f.
- If D (a, b) = 0 then the second derivative test is inconclusive , and the point (a, b) could be any of a minimum , maximum or saddle point .THE TEST FAILS .

1) Find the critical points of f(x,y)= 8x^3-24x y+y^3and points of relative extremum/minimum or saddle points.

In[100]:=
```
f[x_, y_] := 8 x^3 - 24 x y + y^3
```

In[93]:=
```
fx = D[f[x, y], x]
```

Out[93]= $24 x^2 - 24 y$

In[102]:=
```
fxx = D[fx, x]
```

Out[102]= $48 x$

In[94]:=
```
fy = D[f[x, y], y]
```

Out[94]= $-24 x + 3 y^2$

In[103]:=
```
fyy = D[fy, y]
```

Out[103]= $6 y$

In[104]:=
```
fxy = D[fx, y]
```

Out[104]= -24

⋮

In[97]:=
```
Solve[{fx == 0, fy == 0}, {x, y}]
```

Out[97]= $\{\{x \to 0, y \to 0\}, \{x \to 2, y \to 4\}, \{x \to -2(-1)^{1/3}, y \to 4(-1)^{2/3}\}, \{x \to 2(-1)^{2/3}, y \to -4(-1)^{1/3}\}\}$

CRITICAL POINTS ARE → (0, 0) AND (2, 4)

In[106]:=
```
Dis = fxx * fyy - (fxy)^2
```
Out[106]= $-576 + 288\, x\, y$

In[108]:=
```
d1 = Dis /. {x → 0, y → 0}
```
Out[108]= -576

which is less than 0 at (0, 0)

In[109]:=
```
d2 = Dis /. {x → 2, y → 4}
```
Out[109]= 1728

which is greater than 0 at (2, 4)

In[110]:=
```
fxx1 = fxx /. {x → 2, y → 4}
```
Out[110]= 96

which is greater than 0 at (2, 4).

HENCE, RELATIVE MINIMUM IS ATTAINED AT (2,4) BY SECOND PARTIAL DERIVATIVE TEST.

APPLICATIONS AND USES OF MULTIVARIATE CALCULUS

Multivariable calculus can be applied to analyze deterministic systems that have multiple degrees of freedom. Functions with independent variables corresponding to each of the degrees of freedom are often used to model these systems, and multivariable calculus provides tools for characterizing the system dynamics.
Multivariate calculus is used in the optimal control of continuous time dynamic systems. It is used in regression analysis to derive formulas for estimating relationships among various sets of empirical data.Multivariable calculus is used in many fields of natural and social science and engineering to model and study high-dimensional systems that exhibit deterministic behavior.
In economics, for example, consumer choice over a variety of goods, and producer choice over various inputs to use and outputs to produce, are modeled with multivariate calculus. Quantitative analysts in finance also often use multivariate calculus to predict future trends in the stock market.