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# A class of two- and three-level implicit methods of order two in time and four in space based on half-step discretization for two-dimensional fourth order quasi-linear parabolic equations

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ABSTRACT

Novel compact schemes based on half-step discretization are developed to solve two dimensional fourth order quasi-linear parabolic equations. These discretizations use a single computational cell and the boundary conditions are incorporated in a natural way without the use of any fictitious points or discretization. The schemes have second order accuracy in time and fourth order accuracy in space. The proposed methods are directly applicable to problems constituting singular terms. We also present operator splitting method for the solution of linear parabolic equations which allows the use of the one-dimensional tri-diagonal solver multiple times. It is shown that the operator splitting method is unconditionally stable. The numerical schemes have been successfully applied to the two-dimensional Extended Fisher–Kolmogorov equation, Sivashinsky equation, Boussinesq equation and vibrations problem. The illustrative results confirm the theoretical order of magnitude and accuracy of the proposed methods.



## 1. Introduction

Fourth-order parabolic equations arise in a wide class of problems in mathematical physics and engineering such as in vibration of beams and thin plates [27], representing viscoelastic inelastic flows [29], traveling waves in reaction diffusion systems [9], in modeling alloy solidification problem [24] etc. In the past few years, there has been a growing interest in the study of these equations due to their occurrence in a wide variety of application fields, therefore, developing accurate numerical techniques so as to achieve approximate solutions are of great interest.

The purpose of the present paper is to formulate compact high order numerical scheme for the general fourth order nonlinear parabolic partial differential equation (PDE) of the form:

$$A(x,y,t)\nabla^4\phi + \frac{\partial\phi}{\partial t} = f\big(x,y,t,\phi,\phi_x,\phi_y,\nabla^2\phi,\nabla^2\phi_x,\nabla^2\phi_y\big), \quad (x,y,t)\in\Omega\times(0,T], \tag{1}$$

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subject to the initial condition

$$\phi(x,y,0)=\phi_0(x,y),\quad (x,y)\in\Omega, \tag{2}$$

and the boundary conditions

$$\phi=h_1(x,y,t),\ \ \frac{\partial^2\phi}{\partial n^2}=h_2(x,y,t),\ \ (x,y,t)\in\partial\Omega\times(0,T] \tag{3}$$

where $\Omega=\{(x,y):0<x,y<1\}\subset\mathbb{R}^2$, $\nabla^2\phi(x,y,t)\equiv\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}$ represents the two space dimensional Laplacian of the function $\phi(x, y, t)$, $n$ is the outer normal direction to the boundary $\partial\Omega$ and $T$ is the total time. It is assumed that $\phi(x, y, t)\in C^6$ and the coefficient $A\in C^4$ where $C^m$ is the set of all functions whose partial derivatives up to order $m$ are continuous in the region $\Omega\times(0, T]$ in order to maintain the order of accuracy of the difference scheme under consideration. Further, the function $f$ should be at least four times continuously differentiable with respect to all the arguments. The initial data $\phi_0$ should be continuous in the region $\Omega$. Equations of this form are fundamental components of various applied mathematical models as discussed below. The fourth order reaction diffusion equation of the form:

$$\gamma\nabla^4\phi+\frac{\partial\phi}{\partial t}-\nabla^2\phi+\phi^3-\phi=0,\ \ (x,y,t)\in\Omega\times(0,T] \tag{4}$$

where $\gamma$ is a positive constant is a particular model of the type (1)–(3). Eq. (4) with $\gamma=0$ is the standard reaction diffusion equation that is also known as the Fisher–Kolmogorov equation [2]. However, by adding a stabilizing fourth order derivative term to the Fisher–Kolmogorov equation, Dee and Van Saarlos [9] proposed the model expressed by Eq. (4) and termed it as the Extended Fisher–Kolmogorov (EFK) equation. The existence, uniqueness, regularity and other theoretical results for Eq. (4) have been discussed in [7].

In the solidification of dilute binary alloy, a planar solid-liquid interface is commonly observed to be unstable, impulsively assuming a cellular configuration. This state facilitates an asymptotic nonlinear equation to originate which explains the dynamics of the onset and stabilization of cellular configuration given by:

$$\nabla^4\phi+\frac{\partial\phi}{\partial t}-\nabla^2\left(\frac{1}{2}\phi^2-2\phi\right)+\alpha\phi=0,\ \ (x,y,t)\in\Omega\times(0,T] \tag{5}$$

where $\alpha>0$ is a constant (see [24]). The nonlinear Eq. (5) is termed as the Sivashinsky equation which is of fourth order in space.

The following is the other class of two-dimensional fourth order parabolic equation studied in the paper :

$$B(x,y,t)\nabla^4\phi+\frac{\partial^2\phi}{\partial t^2}=g(x,y,t,\phi,\phi_t,\phi_x,\phi_y,\nabla^2\phi,\nabla^2\phi_x,\nabla^2\phi_y),\ \ (x,y,t)\in\Omega\times(0,T], \tag{6}$$

subject to the initial condition (2) and

$$\frac{\partial\phi}{\partial t}(x,y,0)=\phi_1(x,y),\quad (x,y)\in\Omega. \tag{7}$$

where it is assumed that $\phi(x, y, t)\in C^6$, $B\in C^4$ and the function $g$ is of sufficient smoothness and its required higher order derivatives exist in the solution region $\Omega\times(0, T]$. The boundary conditions of second kind given by (3) are considered. The problem arising in the classical theory of vibrating plates of finding the transverse vibrations of a uniform thin square plate with each side of length $L$ hinged at its boundary are directed by the following fourth order PDE [5,27]:

$$\sigma\nabla^4\phi+\mu\frac{\partial^2\phi}{\partial t^2}=0,\ \ 0<x,y<L,\ \ t>0 \tag{8}$$

where $\phi(x, y, t)$ is the vertical deflection at location $(x, y)$ at time $t$, $\sigma>0$ is the flexural rigidity and $\mu>0$ is the mass density per unit area.

The two dimensional Boussinesq equation (BE) has the form

$$\frac{\partial^2\phi}{\partial t^2}=\nabla^2\phi+\alpha\nabla^2\phi^2-\beta\nabla^4\phi,\ \ (x,y)\in\Omega,\ \ t>0 \tag{9}$$

where $\phi(x, y, t)$ is the surface elevation, $\beta>0$ is the dispersion coefficient and $\alpha$ is an amplitude parameter which can be set equal to unity without losing generality. The BE was the first model for the propagation of gravity waves on the surface of water [29] and has various applications, for instance, ion-acoustic wave in plasma, magnetohydrodynamics wave in plasma, longitudinal dispersive wave in elastic rods and pressure wave in liquid-gas bubble mixtures etc. The equation belongs to the KdV family of equations and was first introduced by Boussinesq [3] in 1872.

Another class of two-dimensional fourth order parabolic PDE considered in this study is of the form:

$$\nu^2\nabla^4\phi-2\nu\left(\frac{\partial^3\phi}{\partial x^2\partial t}+\frac{\partial^3\phi}{\partial y^2\partial t}\right)+\frac{\partial^2\phi}{\partial t^2}=d(x,y,t),\ \ (x,y,t)\in\Omega\times(0,T] \tag{10}$$

where $\nu>0$ represents diffusivity. The initial conditions given by (2), (7) and the boundary conditions (3) are considered.

Numerical computation of transverse vibrations of a uniform plate using finite differences have been carried out by a number of authors since 1957. Conte [5] formulated stable difference approximations to a fourth order parabolic equation arising in the study of transverse vibrations of a uniform plate. Twizell and Khaliq [28] developed a finite difference method for the numerical solution of fourth order parabolic PDEs in two space variables using multiderivative method for second order ordinary differential equations. An explicit second order three-layer approximation for the vibrations of a plate with a weak limitation on the stability has been presented in [26]. The operator splitting method is a powerful numerical technique for solving two-dimensional problem wherein the problem is solved in two sweeps while solving the equation implicitly in one dimension in each sweep which leads to substantial decrease in computational effort. Alternating direction implicit (ADI) methods for problems involving the biharmonic operator date back to the work of Conte and Dames [6] and Fairweather and Gourlay [12]. Fairweather [14] formulated second order alternating direction Galerkin methods for approximately solving a linear fourth order PDE governing the transverse vibrations of a thin plate of variable density. Andrade and McKee [1] derived high accuracy splitting formulas for fourth order parabolic equations in one, two and three space dimensions. Witelski and Bowen [30] constructed ADI schemes for the solution of fourth order linear and nonlinear parabolic equations in two dimensions. Recently, Guo and Lü [15] presented parallel difference schemes for two-dimensional fourth order diffusion equation. Danumjaya and Pani [8] proposed fully discrete schemes and derived optimal error estimates for the EFK equation. Khiari and Omrani [18] proposed a Crank-Nicolson type second order convergent finite difference scheme for the two-dimensional EFK equation. Dongdong [10] developed a second order three-level linearly implicit finite difference method for the EFK equation in two-space dimensions. A second order linearized finite difference discretization for the solution of the nonlinear evolutionary Sivashinsky equation was analyzed in [24]. Moeiz and Omrani [25] investigated a linearized three-level difference scheme for the Sivashinsky equation. Related to fourth order equations, Mohanty and Kaur [20–22] proposed three-level and two-level compact implicit variable mesh numerical methods of order two in time and three in space for the solution of one dimensional parabolic PDEs.

Many complexities come up when computing fourth order parabolic equations. Boundary conditions are often not easy to implement and time stepping is perhaps the fundamental issue for time-dependent fourth order problems. Explicit methods for fourth order diffusions have a time step constraint requiring the time step to be of order $h^4$, where $h$ is the grid cell width due to stability requirements whereas the time steps are only restricted to be order $h^2$ for second order diffusions. Due to the time step restriction, the explicit schemes are prohibitive in any meaningful simulation of fourth order parabolic equations and thus implicit schemes are employed. The classical finite difference methods to these equations have second order accuracy in spatial discretization. The accuracy in such a case depends upon the boundary approximation used. A major shortcoming of the classical approach for higher dimensional problems is the widening of the computational stencil as the order of the approximation is increased resulting in increased arithmetic operations. The standard thirteen point approximation of the biharmonic operator leads to the inclusion of fictitious points. Of late, higher order compact difference methods which need less information from neighboring grid points are acquiring popularity owing to their easy implementation and high accuracy in considerably lesser number of nodes. In each spatial dimension, we want a minimum of three grid points for discretizing the highest order derivative terms coming in the problem, hence nine-point discretization scheme is the most appropriate. In this direction, Mohanty et al. [19] proposed compact fourth order method for solving the two-dimensional nonlinear biharmonic equations with third order derivative terms subject to Dirichlet and Neumann boundary conditions. The existing fourth order accurate in space two-level and three-level implicit difference formulas of [23] for the numerical solution of two-space dimensional fourth order parabolic problems of second kind use nine spatial grid points but these methods fail at singular points for problems in polar coordinates, so a special treatment was required to solve singular problems leading to tedious calculations. In this paper, using the same number of grid points, numerical schemes based on half-step discretization are developed to solve fourth order evolution equations in two-space dimensions. This new approach involves half-step points just as in case of the discretization developed by Chawla and Shivakumar [4] for a two-point boundary-value problem and hence is referred to as half-step discretization.

In this paper, we construct high order compact schemes for parabolic PDEs in two dimensions that are given in terms of fourth order spatial operators and are first or second order with respect to time which are directly applicable to singular problems. In Section 2, we present and derive the two-level implicit scheme for fourth order nonlinear time dependent problems (1)–(3). A three-level implicit finite difference algorithm for solving another class of fourth order parabolic PDE in two dimensions is proposed in Section 3 involving nine spatial grid points of a single compact cell. In Section 4, stability of the proposed two-level method for a linear problem is discussed. Further, in this section, the two-step operator splitting method is developed for a fourth order linear parabolic equation with mixed terms of time derivatives of the Laplacian and using von Neumann stability analysis it is proved to be unconditionally stable. Finally, in Section 5, we employ some numerical examples on problems of physical significance to corroborate the accuracy of the proposed methods. Some concluding remarks are given in Section 6.

## 2. Two level implicit compact scheme

### 2.1. Formulation and derivation

In this section, we derive a new difference formula which is fourth order accurate in space to the general fourth order PDE (1).

Divide the domain of definition $\Omega \times (0, T]$ by parallel lines with the space mesh step $\Delta x = \Delta y = h > 0$ and $k > 0$ is the time mesh step. Each grid point is denoted by $(x_l, y_m, t_j)$ where $x_l = lh,\ y_m = mh,\ l, m = 0, 1, \ldots, N+1,\ t_j = jk,\ j = 0, 1, \ldots, J$ with $(N+1)h = 1$ and $Jk = T$. The neighboring off-step points are given by $(x_{l\pm\frac{1}{2}}, y_m, t_j),\ (x_l, y_{m\pm\frac{1}{2}}, t_j)$ where $x_{l\pm1/2} = x_l \pm \frac{h}{2},\ y_{m\pm1/2} = y_m \pm \frac{h}{2},\ l, m = 1, \ldots, N$. The mesh ratio parameter is given by $\lambda = k/h^2$. Let the exact solution values of $\phi$ and $\psi$ at the grid point $(x_l, y_m, t_j)$ be denoted by $\Phi^j_{l,m}$ and $\Psi^j_{l,m}$, respectively while $\phi^j_{l,m}$ and $\psi^j_{l,m}$ denote their approximate values.

Note that for $\phi(x, y, t)$, the initial and boundary conditions are given by (2) and (3). Since the grid lines are parallel to the coordinate axes and the value of $\phi$ is exactly known at the boundary, this implies that the successive tangential partial derivatives are also known there. For instance, at the plane $x = 0$, the values of $\phi$ and $\phi_{xx}$ are known, hence the values of $\phi_y$, $\phi_{yy}$, etc. are also known which is sufficient to determine $\nabla^2\phi$ at $x = 0$.

We reduce the original problems (1)–(3) into an equivalent problem of second order equations by introducing $\psi = \nabla^2\phi$ to obtain

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \psi(x, y, t), \quad (x, y) \in \Omega \tag{11.1}$$

$$A(x, y, t)\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + \frac{\partial\phi}{\partial t} = f(x, y, t, \phi, \phi_x, \phi_y, \psi, \psi_x, \psi_y), \quad (x, y, t) \in \Omega \times (0, T]. \tag{11.2}$$

subject to the initial condition (2) and Dirichlet boundary conditions

$$\phi = h_1(x, y, t), \quad \psi = h_3(x, y, t), \quad (x, y, t) \in \partial\Omega \times (0, T] \tag{12}$$

Systems (11.1) and (11.2) represent a splitting of (1) into two coupled equations.

Let $\bar{t}_j = t_j + \theta k,\ \theta \in (0, 1)$ is a parameter to be determined. We then apply the $\theta$-weighted scheme to the solution variables $\phi$ and $\psi$. In order to achieve high accuracy method, the following approximations to the solution variables and their partial derivatives are needed. For $S = \Phi,\ \Psi$ and for $p, q = 0, \pm1$, we let:

$$\bar{S}^j_{l+p,m+q} = \theta S^{j+1}_{l+p,m+q} + (1-\theta)S^j_{l+p,m+q}, \tag{13.1}$$

$$\bar{S}^j_{l\pm\frac{1}{2},m} = \left(\bar{S}^j_{l\pm1,m} + \bar{S}^j_{l,m}\right)/2, \tag{13.2}$$

$$\bar{S}^j_{l,m\pm\frac{1}{2}} = \left(\bar{S}^j_{l,m\pm1} + \bar{S}^j_{l,m}\right)/2, \tag{13.3}$$

$$\bar{S}^j_{t_{l+p,m+q}} = \left(S^{j+1}_{l+p,m+q} - S^j_{l+p,m+q}\right)/k, \tag{13.4}$$

$$\bar{S}^j_{t_{l\pm\frac{1}{2},m}} = \left(S^{j+1}_{l\pm1,m} + S^{j+1}_{l,m} - S^j_{l\pm1,m} - S^j_{l,m}\right)/2k, \tag{13.5}$$

$$\bar{S}^j_{t_{l,m\pm\frac{1}{2}}} = \left(S^{j+1}_{l,m\pm1} + S^{j+1}_{l,m} - S^j_{l,m\pm1} - S^j_{l,m}\right)/2k, \tag{13.6}$$

$$\bar{S}^j_{x_{l,m}} = \left(\bar{S}^j_{l+1,m} - \bar{S}^j_{l-1,m}\right)/2h, \tag{13.7}$$

$$\bar{S}^j_{x_{l\pm\frac{1}{2},m}} = \pm\left(\bar{S}^j_{l\pm1,m} - \bar{S}^j_{l,m}\right)/h, \tag{13.8}$$

$$\bar{S}^j_{x_{l,m\pm\frac{1}{2}}} = \left(\bar{S}^j_{l+1,m\pm1} - \bar{S}^j_{l-1,m\pm1} + \bar{S}^j_{l+1,m} - \bar{S}^j_{l-1,m}\right)/4h, \tag{13.9}$$

$$\bar{S}^j_{y_{l,m}} = \left(\bar{S}^j_{l,m+1} - \bar{S}^j_{l,m-1}\right)/2h, \tag{13.10}$$

$$\bar{S}^j_{y_{l\pm\frac{1}{2},m}} = \left(\bar{S}^j_{l\pm1,m+1} - \bar{S}^j_{l\pm1,m-1} + \bar{S}^j_{l,m+1} - \bar{S}^j_{l,m-1}\right)/4h, \tag{13.11}$$

$$\bar{S}^j_{y_{l,m\pm\frac{1}{2}}} = \pm\left(\bar{S}^j_{l,m\pm1} - \bar{S}^j_{l,m}\right)/h, \tag{13.12}$$

$$\bar{S}^j_{xx_{l,m+q}} = \left(\bar{S}^j_{l+1,m+q} - 2\bar{S}^j_{l,m+q} + \bar{S}^j_{l-1,m+q}\right)/h^2, \tag{13.13}$$

$$\overline{S}^j_{yy_{l+p,m}} = \left(\overline{S}^j_{l+p,m+1} - 2\overline{S}^j_{l+p,m} + \overline{S}^j_{l+p,m-1}\right)\Big/h^2. \tag{13.14}$$

The functional approximations associated with off-step nodal points are defined as:

$$\overline{F}^j_{l\pm\frac{1}{2},m} = f\left(x_{l\pm\frac{1}{2}}, y_m, \bar{t}_j, \overline{\Phi}^j_{l\pm\frac{1}{2},m}, \overline{\Phi}^j_{x_{l\pm\frac{1}{2},m}}, \overline{\Phi}^j_{y_{l\pm\frac{1}{2},m}}, \overline{\Psi}^j_{l\pm\frac{1}{2},m}, \overline{\Psi}^j_{x_{l\pm\frac{1}{2},m}}, \overline{\Psi}^j_{y_{l\pm\frac{1}{2},m}}\right), \tag{14.1}$$

$$\overline{F}^j_{l,m\pm\frac{1}{2}} = f\left(x_l, y_{m\pm\frac{1}{2}}, \bar{t}_j, \overline{\Phi}^j_{l,m\pm\frac{1}{2}}, \overline{\Phi}^j_{x_{l,m\pm\frac{1}{2}}}, \overline{\Phi}^j_{y_{l,m\pm\frac{1}{2}}}, \overline{\Psi}^j_{l,m\pm\frac{1}{2}}, \overline{\Psi}^j_{x_{l,m\pm\frac{1}{2}}}, \overline{\Psi}^j_{y_{l,m\pm\frac{1}{2}}}\right). \tag{14.2}$$

In addition, the following combinations are needed:

$$\overline{\overline{\Phi}}^j_{l,m} = \overline{\Phi}^j_{l,m} + h^2\left(a_1\overline{\Phi}^j_{xx_{l,m}} + a_2\overline{\Phi}^j_{yy_{l,m}}\right), \tag{15.1}$$

$$\overline{\overline{\Psi}}^j_{l,m} = \overline{\Psi}^j_{l,m} + h^2\left(b_1\overline{\Psi}^j_{xx_{l,m}} + b_2\overline{\Psi}^j_{yy_{l,m}}\right), \tag{15.2}$$

$$\overline{\overline{\Phi}}^j_{x_{l,m}} = \overline{\Phi}^j_{x_{l,m}} + h\left[a_3\left(\overline{\Psi}^j_{l+1,m} - \overline{\Psi}^j_{l-1,m}\right) + a_4\left(\overline{\Phi}^j_{yy_{l+1,m}} - \overline{\Phi}^j_{yy_{l-1,m}}\right)\right], \tag{15.3}$$

$$\overline{\overline{\Phi}}^j_{y_{l,m}} = \overline{\Phi}^j_{y_{l,m}} + h\left[a_5\left(\overline{\Psi}^j_{l,m+1} - \overline{\Psi}^j_{l,m-1}\right) + a_6\left(\overline{\Phi}^j_{xx_{l,m+1}} - \overline{\Phi}^j_{xx_{l,m-1}}\right)\right], \tag{15.4}$$

$$\overline{\overline{\Psi}}^j_{x_{l,m}} = \overline{\Psi}^j_{x_{l,m}} + h\left[b_3\left(\overline{F}^j_{l+\frac{1}{2},m} - \overline{F}^j_{l-\frac{1}{2},m}\right) + b_4\left(\overline{\Phi}^j_{t_{l+1,m}} - \overline{\Phi}^j_{t_{l-1,m}}\right) + b_5\left(\overline{\Psi}^j_{yy_{l+1,m}} - \overline{\Psi}^j_{yy_{l-1,m}}\right)\right] + h^2\left(b_6\overline{\Psi}^j_{xx_{l,m}} + b_7\overline{\Psi}^j_{yy_{l,m}}\right), \tag{15.5}$$

$$\overline{\overline{\Psi}}^j_{y_{l,m}} = \overline{\Psi}^j_{y_{l,m}} + h\left[b_8\left(\overline{F}^j_{l,m+\frac{1}{2}} - \overline{F}^j_{l,m-\frac{1}{2}}\right) + b_9\left(\overline{\Phi}^j_{t_{l,m+1}} - \overline{\Phi}^j_{t_{l,m-1}}\right) + b_{10}\left(\overline{\Psi}^j_{xx_{l,m+1}} - \overline{\Psi}^j_{xx_{l,m-1}}\right)\right] + h^2\left(b_{11}\overline{\Psi}^j_{xx_{l,m}} + b_{12}\overline{\Psi}^j_{yy_{l,m}}\right), \tag{15.6}$$

where $a_i$, $1 \le i \le 6$ and $b_i$, $1 \le i \le 12$ are parameters to be suitably determined.

Finally, we let

$$\overline{\overline{F}}^j_{l,m} = f\left(x_l, y_m, \bar{t}_j, \overline{\overline{\Phi}}^j_{l,m}, \overline{\overline{\Phi}}^j_{x_{l,m}}, \overline{\overline{\Phi}}^j_{y_{l,m}}, \overline{\overline{\Psi}}^j_{l,m}, \overline{\overline{\Psi}}^j_{x_{l,m}}, \overline{\overline{\Psi}}^j_{y_{l,m}}\right). \tag{16}$$

The PDEs (11.1) and (11.2) at the grid point $(x_l, y_m, t_j)$, $l, m = 1, \dots, N,\ j = 0, 1, \dots$ are discretized as:

$$L[\overline{\phi}] \equiv \left[\delta_x^2 + \delta_y^2 + \frac{1}{6}\delta_x^2\delta_y^2\right]\overline{\Phi}^j_{l,m} = \frac{h^2}{12}\left[\overline{\Psi}^j_{l+1,m} + \overline{\Psi}^j_{l-1,m} + \overline{\Psi}^j_{l,m+1} + \overline{\Psi}^j_{l,m-1} + 8\overline{\Psi}^j_{l,m}\right] + \overline{T}^{j^{(1)}}_{l,m}, \tag{17.1}$$

$$\begin{aligned} L[\overline{\psi}] &\equiv \left[I_1\left(\delta_x^2 + \delta_y^2\right) + I_2\delta_x^2\delta_y^2\right]\overline{\Psi}^j_{l,m} \\ &= -\frac{h^2}{12}\left[R_1\overline{\Phi}^j_{t_{l+1,m}} + R_2\overline{\Phi}^j_{t_{l-1,m}} + R_3\overline{\Phi}^j_{t_{l,m+1}} + R_4\overline{\Phi}^j_{t_{l,m-1}} + 8\overline{\Phi}^j_{t_{l,m}}\right] \\ &\quad + \frac{h^2}{3}\left[J_1\overline{F}^j_{l+\frac{1}{2},m} + J_2\overline{F}^j_{l-\frac{1}{2},m} + J_3\overline{F}^j_{l,m+\frac{1}{2}} + J_4\overline{F}^j_{l,m-\frac{1}{2}} - \overline{\overline{F}}^j_{l,m}\right] + \overline{T}^{j^{(2)}}_{l,m} \end{aligned} \tag{17.2}$$

where $\delta_x\Phi_{l,m} = \Phi_{l+\frac{1}{2},m} - \Phi_{l-\frac{1}{2},m}$, $\delta_y\Phi_{l,m} = \Phi_{l,m+\frac{1}{2}} - \Phi_{l,m-\frac{1}{2}}$ are central difference operators with respect to $x$- and $y$- directions, respectively; the local truncation errors $\overline{T}^{j^{(1)}}_{l,m} = O(k^2h^2 + kh^4 + h^6)$, $\overline{T}^{j^{(2)}}_{l,m} = O(k^2h^2 + kh^4 + h^6)$ and

$$I_1 = \overline{A}^j_{l,m} + \frac{h^2}{12}\left(\overline{A}^j_{xx_{l,m}} + \overline{A}^j_{yy_{l,m}}\right) - \frac{h^2}{6\overline{A}^j_{l,m}}\left(\overline{A}^{j^2}_{xl,m} + \overline{A}^{j^2}_{yl,m}\right), \quad I_2 = \frac{1}{6}\overline{A}^j_{l,m},$$

$$R_1 = \left(1 - \frac{h\overline{A}^j_{x_{l,m}}}{\overline{A}^j_{l,m}}\right),\ R_2 = \left(1 + \frac{h\overline{A}^j_{x_{l,m}}}{\overline{A}^j_{l,m}}\right),\ R_3 = \left(1 - \frac{h\overline{A}^j_{y_{l,m}}}{\overline{A}^j_{l,m}}\right),\ R_4 = \left(1 + \frac{h\overline{A}^j_{y_{l,m}}}{\overline{A}^j_{l,m}}\right),$$

$$J_1 = \left(1 - \frac{h\overline{A}^j_{x_{l,m}}}{2\overline{A}^j_{l,m}}\right),\ J_2 = \left(1 + \frac{h\overline{A}^j_{x_{l,m}}}{2\overline{A}^j_{l,m}}\right),\ J_3 = \left(1 - \frac{h\overline{A}^j_{y_{l,m}}}{2\overline{A}^j_{l,m}}\right),\ J_4 = \left(1 + \frac{h\overline{A}^j_{y_{l,m}}}{2\overline{A}^j_{l,m}}\right),$$

$\overline{A}^j_{l,m} = A(x_l, y_m, \bar{t}_j)$, $\overline{A}^j_{x_{l,m}} = A_x(x_l, y_m, \bar{t}_j)$, $\overline{A}^j_{xx_{l,m}} = A_{xx}(x_l, y_m, \bar{t}_j)$ etc.

The formulas (17.1) and (17.2) are free from the terms $1/x_{l\pm1}$ and $1/y_{m\pm1}$, hence solvable for $l, m = 1, \dots, N$ for $0 < x$, $y < 1, t > 0$. The proposed method is directly applicable to singular problems since it involves terms like $1/x_{l-1/2}$ instead of $1/x_{l-1}$ without requiring any fictitious points or particular treatment at the point of singularity. Eventually, computational complexity has been reduced. The use of half-step nodal points enables the direct application of the schemes (17.1) and (17.2) to singular problems.

For the derivation of the method, we follow the technique of Chawla and Shivakumar [4]. At the grid point $(x_l, y_m, t_j)$, we denote

$$W_{abc} = \left(\frac{\partial^{a+b+c}W}{\partial x^a \partial y^b \partial t^c}\right)_{l,m}^j, \quad W = A, B, \Phi, \Psi;\ a, b, c = 0, 1, 2, \dots, \tag{18.1}$$

$$\alpha_{l,m}^j = \left(\frac{\partial f}{\partial \Phi}\right)_{l,m}^j, \beta_{l,m}^j = \left(\frac{\partial f}{\partial \Psi}\right)_{l,m}^j, \gamma_{l,m}^j = \left(\frac{\partial f}{\partial \Phi_x}\right)_{l,m}^j, \mu_{l,m}^j = \left(\frac{\partial f}{\partial \Psi_x}\right)_{l,m}^j, \nu_{l,m}^j = \left(\frac{\partial f}{\partial \Phi_y}\right)_{l,m}^j,$$
$$\omega_{l,m}^j = \left(\frac{\partial f}{\partial \Psi_y}\right)_{l,m}^j, \xi_{l,m}^j = \left(\frac{\partial f}{\partial t}\right)_{l,m}^j. \tag{18.2}$$

Using Taylor series expansion and differentiating equations (11.1) and (11.2) twice with respect to $x$ and $y$, we get

$$L[\Phi] = \frac{h^2}{12}\left[\Psi_{l+1,m}^j + \Psi_{l-1,m}^j + \Psi_{l,m+1}^j + \Psi_{l,m-1}^j + 8\Psi_{l,m}^j\right] + O(h^6), \tag{19.1}$$

$$L[\Psi] = -\frac{h^2}{12}\left[R_1\Phi_{t_{l+1,m}}^j + R_2\Phi_{t_{l-1,m}}^j + R_3\Phi_{t_{l,m+1}}^j + R_4\Phi_{t_{l,m-1}}^j + 8\Phi_{t_{l,m}}^j\right]$$
$$+\frac{h^2}{3}\left[J_1F_{l+\frac{1}{2},m}^j + J_2F_{l-\frac{1}{2},m}^j + J_3F_{l,m+\frac{1}{2}}^j + J_4F_{l,m-\frac{1}{2}}^j - F_{l,m}^j +\right] + O(h^6), \tag{19.2}$$

where $L[\Phi]$, $L[\Psi]$ may be obtained from $L[\overline{\Phi}]$, $L[\overline{\Psi}]$, respectively, by replacing $\bar{t}_j$ by $t_j$, $\overline{\Phi}$ by $\Phi$ and $\overline{\Psi}$ by $\Psi$.

With the help of approximations (13.1)–(13.14), simplifying (14.1) and (14.2), we obtain

$$\overline{F}_{l\pm\frac{1}{2},m}^j = F_{l\pm\frac{1}{2},m}^j + \theta k T_1 + \frac{h^2}{24}T_2 \pm O(kh + h^3), \tag{20.1}$$

$$\overline{F}_{l,m\pm\frac{1}{2}}^j = F_{l,m\pm\frac{1}{2}}^j + \theta k T_1 + \frac{h^2}{24}T_3 \pm O(kh + h^3), \tag{20.2}$$

where

$$T_1 = \Phi_{001}\alpha_{l,m}^j + \Psi_{001}\beta_{l,m}^j + \Phi_{101}\gamma_{l,m}^j + \Psi_{101}\mu_{l,m}^j + \Phi_{011}\nu_{l,m}^j + \Psi_{011}\omega_{l,m}^j + \xi_{l,m}^j,$$
$$T_2 = 3\Phi_{200}\alpha_{l,m}^j + 3\Psi_{200}\beta_{l,m}^j + \Phi_{300}\gamma_{l,m}^j + \Psi_{300}\mu_{l,m}^j + (3\Phi_{210} + 4\Phi_{030})\nu_{l,m}^j + (3\Psi_{210} + 4\Psi_{030})\omega_{l,m}^j,$$
$$T_3 = 3\Phi_{020}\alpha_{l,m}^j + 3\Psi_{020}\beta_{l,m}^j + \Phi_{030}\nu_{l,m}^j + \Psi_{030}\omega_{l,m}^j + (3\Phi_{120} + 4\Phi_{300})\gamma_{l,m}^j + (3\Psi_{120} + 4\Psi_{300})\mu_{l,m}^j.$$

Using the approximations (13.1)–(13.14) and (20.1) and (20.2), we obtain

$$\overline{\overline{\Phi}}_{l,m}^j = \Phi_{l,m}^j + \theta k\Phi_{001} + \frac{h^2}{6}T_4 + O(k^2 + kh^2 + h^4), \tag{21.1}$$

$$\overline{\overline{\Psi}}_{l,m}^j = \Psi_{l,m}^j + \theta k\Psi_{001} + \frac{h^2}{6}T_5 + O(k^2 + kh^2 + h^4), \tag{21.2}$$

$$\overline{\overline{\Phi}}_{x_{l,m}}^j = \Phi_{x_{l,m}}^j + \theta k\Phi_{101} + \frac{h^2}{6}T_6 + O(k^2 + kh^2 + h^4), \tag{21.3}$$

$$\overline{\overline{\Phi}}_{y_{l,m}}^j = \Phi_{y_{l,m}}^j + \theta k\Phi_{011} + \frac{h^2}{6}T_7 + O(k^2 + kh^2 + h^4), \tag{21.4}$$

$$\overline{\overline{\Psi}}_{x_{l,m}}^j = \Psi_{x_{l,m}}^j + \theta k\Psi_{101} + \frac{h^2}{6}T_8 + O(k^2 + kh^2 + h^4), \tag{21.5}$$

$$\overline{\overline{\Psi}}_{y_{l,m}}^j = \Psi_{y_{l,m}}^j + \theta k\Psi_{011} + \frac{h^2}{6}T_9 + O(k^2 + kh^2 + h^4), \tag{21.6}$$

for

$$T_4 = 6(a_1\Phi_{200} + a_2\Phi_{020}),$$
$$T_5 = 6(b_1\Psi_{200} + b_2\Psi_{020}),$$
$$T_6 = (1 + 12a_3)\Phi_{300} + 12(a_3 + a_4)\Phi_{120},$$
$$T_7 = (1 + 12a_5)\Phi_{030} + 12(a_5 + a_6)\Phi_{210},$$
$$T_8 = \left(1 + 6b_3\overline{A}^j_{l,m}\right)\Psi_{300} + 6\left(b_3\overline{A}^j_{l,m} + 2b_5\right)\Psi_{120} + 6\left(b_3\overline{A}^j_{x_{l,m}} + b_6\right)\Psi_{200} + 6\left(b_3\overline{A}^j_{x_{l,m}} + b_7\right)\Psi_{020} + 6(b_3 + 2b_4)\Psi_{101},$$
$$T_9 = \left(1 + 6b_8\overline{A}^j_{l,m}\right)\Psi_{030} + 6\left(b_8\overline{A}^j_{l,m} + 2b_{10}\right)\Psi_{210} + 6\left(b_8\overline{A}^j_{y_{l,m}} + b_{11}\right)\Psi_{020} + 6\left(b_8\overline{A}^j_{y_{l,m}} + b_{12}\right)\Psi_{200} + 6(b_8 + 2b_9)\Psi_{011}.$$

Invoking Taylor series expansion and using (21.1)–(21.6) in (16), we obtain the estimation in its expanded form:

$$\overline{\overline{F}}^j_{l,m} = F^j_{l,m} + \theta k T_1 + \frac{h^2}{6}T_{10} + O(k^2 + kh^2 + h^4), \tag{22}$$

where

$$T_{10} = T_4\alpha^j_{l,m} + T_5\beta^j_{l,m} + T_6\gamma^j_{l,m} + T_7\nu^j_{l,m} + T_8\mu^j_{l,m} + T_9\omega^j_{l,m},$$
$$F^j_{l,m} \equiv f\left(x_l, y_m, t_j, \phi^j_{l,m}, \phi^j_{x_{l,m}}, \phi^j_{y_{l,m}}, \psi^j_{l,m}, \psi^j_{x_{l,m}}, \psi^j_{y_{l,m}}\right).$$

Further using the approximations (13.1)–(13.14) and $\overline{A}^j_{l,m} = A^j_{l,m} + \theta k A^j_{t_{l,m}} + O(k^2)$, etc. we may write

$$L[\overline{\Phi}] = L[\Phi] + \theta k h^2(\Phi_{201} + \Phi_{021}) + O(kh^4 + k^2h^2), \tag{23.1}$$

$$L[\overline{\Psi}] = L[\Psi] + \theta k h^2 A_{001}(\Phi_{200} + \Phi_{020}) + \theta k h^2 A_{000}(\Psi_{201} + \Psi_{021}) + O(kh^4 + k^2h^2). \tag{23.2}$$

Differentiating (11.1) w.r.t '$t$' at the grid point $(x_l, y_m, t_j)$ and with the help of relations (19.1) and (23.1), the local truncation error associated with (17.1) is obtained as $\overline{T}^{j(1)}_{l,m} = O(k^2h^2 + kh^4 + h^6)$. Using the relations (19.2), (23.2) and the approximations (20.1) and (20.2), (22) in the right-hand side of (17.2), the local truncation error associated with (17.2) is obtained as

$$\overline{T}^{j(2)}_{l,m} = kh^2\left(\theta - \frac{1}{2}\right)\Phi_{002} + \frac{h^4}{36}(T_2 + T_3 - 2T_{10}) + O(k^2h^2 + kh^4 + h^6) \tag{24}$$

The proposed difference method (17.2) is of $O(k^2 + kh^2 + h^4)$ if the coefficients of $kh^2$ and $h^4$ in (24) are zero, hence we obtain $\theta = 1/2$ and

$$T_2 + T_3 = 2T_{10},$$

which gives

$$a_1 = a_2 = \frac{1}{4},\quad a_3 = a_5 = \frac{1}{8},\quad a_4 = a_6 = 0,$$

$$b_1 = b_2 = \frac{1}{4},\quad b_3 = b_8 = \frac{1}{4\overline{A}^j_{l,m}},\quad b_4 = b_9 = -\frac{1}{8\overline{A}^j_{l,m}},\quad b_5 = b_{10} = 0,\quad b_6 = b_7 = -\frac{\overline{A}^j_{x_{l,m}}}{4\overline{A}^j_{l,m}},\quad b_{11} = b_{12} = -\frac{\overline{A}^j_{y_{l,m}}}{4\overline{A}^j_{l,m}}$$

Hence for the above set of parameters, $\overline{T}^{j(2)}_{l,m} = O(k^2h^2 + kh^4 + h^6)$ and thus the difference schemes (17.1) and (17.2) are of $O(k^2 + kh^2 + h^4)$.

Schemes (17.1) and (17.2) lead to the block tri-diagonal matrix structure and thus conveniently solved using standard algorithms, block Gauss–Seidel iterative method for linear discrete equations while the nonlinear difference equations are treated by Newton's non-linear block iteration method [16]. The difference formulas are fourth order accurate for the fixed value of the mesh ratio parameter $\lambda = k/h^2$.

Let us now consider the two-space dimensional quasi-linear initial boundary value problems (1)–(3) wherein the coefficient $A \equiv A(x, y, t, \phi, \nabla^2\phi)$. In this case, we use the following central difference approximations in (17.1) and (17.2):

$$\overline{A}^j_{x_{l,m}} = \left(\overline{A}^j_{l+1,m} - \overline{A}^j_{l-1,m}\right)\Big/(2h) + O(h^2), \tag{25.1}$$

$$\overline{A}^j_{y_{l,m}} = \left(\overline{A}^j_{l,m+1} - \overline{A}^j_{l,m-1}\right)\Big/(2h) + O(h^2), \tag{25.2}$$

$$\overline{A}^j_{xx_{l,m}} = \left(\overline{A}^j_{l+1,m} - 2\overline{A}^j_{l,m} + \overline{A}^j_{l-1,m}\right)\Big/h^2 + O(h^2), \tag{25.3}$$

$$\overline{A}^j_{yy_{l,m}} = \left(\overline{A}^j_{l,m+1} - 2\overline{A}^j_{l,m} + \overline{A}^j_{l,m-1}\right)\big/h^2 + O(h^2), \tag{25.4}$$

where for $c = 0, \pm 1,$ we have

$$\overline{A}^j_{l+c,m} = A\left(x_{l+c}, y_m, \bar{t}_j, \overline{\Phi}^j_{l+c,m}, \overline{\Psi}^j_{l+c,m}\right),$$

$$\overline{A}^j_{l,m+c} = A\left(x_l, y_{m+c}, \bar{t}_j, \overline{\Phi}^j_{l,m+c}, \overline{\Psi}^j_{l,m+c}\right).$$

Using the above central differences in (17.1)and(17.2), the order of the difference scheme is retained and thus we obtain $O(k^2 + kh^2 + h^4)$ difference method for the two-dimensional fourth order quasi-linear problem.

### 2.2. Stability consideration

We consider the two space dimensional fourth order linear parabolic PDE of the form

$$\nabla^4\phi + \frac{\partial\phi}{\partial t} = f(x, y, t), \quad 0 < x, y < 1, \quad t > 0 \tag{26}$$

The two-level difference formulas (17.1) and (17.2) when applied to the differential equation (26) can be given the matrix notation as

$$\boldsymbol{P}\boldsymbol{u}^{j+1} = (-\boldsymbol{P} + \boldsymbol{Q})\boldsymbol{u}^j + \boldsymbol{r}, \tag{27}$$

where

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P_1} & -h^2\boldsymbol{P_2} \\ \boldsymbol{P_2} & \frac{\lambda}{2}\boldsymbol{P_1} \end{bmatrix}_{2N^2\times 2N^2}, \quad \boldsymbol{Q} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ 2\boldsymbol{P_2} & \boldsymbol{0} \end{bmatrix}_{2N^2\times 2N^2}, \quad \boldsymbol{u} = \begin{bmatrix} \boldsymbol{\phi} \\ \boldsymbol{\psi} \end{bmatrix}, \quad \boldsymbol{r} = \begin{bmatrix} \boldsymbol{r_1} \\ \boldsymbol{r_2} \end{bmatrix}.$$

$\boldsymbol{P_1}$ and $\boldsymbol{P_2}$ are $N^2 \times N^2$ tri-block diagonal matrices given by

$$\boldsymbol{P_1} = [\boldsymbol{C}, \boldsymbol{D}, \boldsymbol{C}], \ \boldsymbol{P_2} = [\boldsymbol{E}, \boldsymbol{F}, \boldsymbol{E}],$$

for

$$\boldsymbol{C} = 2[1, 4, 1], \ \boldsymbol{D} = 2[4, -20, 4], \ \boldsymbol{E} = [0, 1, 0], \ \boldsymbol{F} = [1, 8, 1]$$

where we denote

$$[a, b, c] = \begin{bmatrix} b & c & 0 & \cdots & 0 \\ a & b & c & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & a & b & c \\ 0 & \cdots & 0 & a & b \end{bmatrix}$$

to be the $N \times N$ tri-diagonal matrix having eigenvalues $b + 2\sqrt{ac}\cos(2\phi),\ 2\phi = (s\pi)/(N+1),\ s = 1(1)N$; $\boldsymbol{\phi}, \boldsymbol{\psi}$ are solution vectors and $\boldsymbol{r_1}, \boldsymbol{r_2}$ are column vectors of order $N^2$ consisting of homogenous functions, initial and boundary values of the block system (27).

The eigenvalues of $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$ are given by $-48(\sin^2\phi_1 + \sin^2\phi_2) + 32\sin^2\phi_1\sin^2\phi_2$ and $12 - 4(\sin^2\phi_1 + \sin^2\phi_2)$, respectively, where $\phi_1 = (\pi r)/(N+1),\ r = 1, 2, \ldots, N$ and $\phi_2 = (\pi s)/(N+1),\ s = 1, 2, \ldots, N$.

For stability of differential equation (26), we consider the homogenous part of the difference scheme (27), which may be written as

$$\boldsymbol{u}^{j+1} = (-\boldsymbol{I} + \boldsymbol{P}^{-1}\boldsymbol{Q})\,\boldsymbol{u}^j. \tag{28}$$

We denote $\boldsymbol{\varepsilon}^j = \boldsymbol{u}^j - \boldsymbol{U}^j$ as the error vector at the $j$th iterate (in the absence of round-off errors), where

$$U^j = \begin{bmatrix} \Phi \\ \Psi \end{bmatrix}^j,$$

$\Phi$ and $\Psi$ being exact solution vectors. We may write the error equation as:

$$\boldsymbol{\varepsilon}^{j+1} = \boldsymbol{H}\boldsymbol{\varepsilon}^j,$$

where the amplification matrix $\boldsymbol{H}$ is given by $\boldsymbol{H} = -\boldsymbol{I} + \boldsymbol{P}^{-1}\boldsymbol{Q}$.

The eigenvalues $\zeta$ of matrix $\boldsymbol{P}$ satisfies the characteristic equation

$$\det\begin{bmatrix} -48(\sin^2\phi_1 + \sin^2\phi_2) + 32\sin^2\phi_1\sin^2\phi_2 - \zeta & -h^2(12 - 4(\sin^2\phi_1 + \sin^2\phi_2)) \\ 12 - 4(\sin^2\phi_1 + \sin^2\phi_2) & -24\lambda(\sin^2\phi_1 + \sin^2\phi_2) + 16\lambda\sin^2\phi_1\sin^2\phi_2 - \zeta \end{bmatrix} = 0, \tag{29}$$

Further, 0 is the only eigenvalue of $\boldsymbol{Q}$. Let $\rho$ be the eigenvalue of $\boldsymbol{P}^{-1}\boldsymbol{Q}$, where $\zeta$ and 0 are the eigenvalues of $\boldsymbol{P}$ and $\boldsymbol{Q}$, respectively. Then $-1 + \rho$ is the eigenvalue of $\boldsymbol{H}$. Hence the scheme is stable as long as $0 < \rho < 2$.

## 3. Three level implicit compact scheme

In this section, we present high order compact difference formula to solve fourth order parabolic PDE (6).

By introducing $\psi = \nabla^2\phi$, Eq. (6) may be rewritten as two simultaneous PDEs of the form

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \psi(x,y,t), \quad (x,y) \in \Omega, \tag{30.1}$$

$$B(x,y,t)\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + \frac{\partial^2\phi}{\partial t^2} = g(x,y,t,\phi,\phi_t,\phi_x,\phi_y,\psi,\psi_x,\psi_y), \ (x,y,t) \in \Omega\times(0,T], \tag{30.2}$$

subject to the initial conditions (2) and (7). Using the given boundary conditions (3) and successive tangential partial derivative values, $\phi$ and $\psi$ are known on all sides of the solution domain $\Omega\times(0,T]$.

We consider the following approximations: for $r,s = 0,\pm1,$ we denote

$$\widehat{\Phi}^j_{l+r,m+s} = \tau\Phi^{j+1}_{l+r,m+s} + (1-2\tau)\Phi^j_{l+r,m+s} + \tau\Phi^{j-1}_{l+r,m+s}, \ \ 0<\tau<1, \tag{31.1}$$

$$\widehat{\Phi}^j_{t_{l+r,m+s}} = \left(\Phi^{j+1}_{l+r,m+s} - \Phi^{j-1}_{l+r,m+s}\right)/2k, \tag{31.2}$$

$$\widehat{\Phi}^j_{t_{l\pm\frac{1}{2},m}} = \left(\Phi^{j+1}_{l\pm1,m} + \Phi^{j+1}_{l,m} - \Phi^{j-1}_{l\pm1,m} - \Phi^{j-1}_{l,m}\right)/4k, \tag{31.3}$$

$$\widehat{\Phi}^j_{t_{l,m\pm\frac{1}{2}}} = \left(\Phi^{j+1}_{l,m\pm1} + \Phi^{j+1}_{l,m} - \Phi^{j-1}_{l,m\pm1} - \Phi^{j-1}_{l,m}\right)/4k, \tag{31.4}$$

$$\widehat{\Phi}^j_{tt_{l+r,m+s}} = \left(\Phi^{j+1}_{l+r,m+s} - 2\Phi^j_{l+r,m+s} + \Phi^{j-1}_{l+r,m+s}\right)/k^2, \tag{31.5}$$

$$\widehat{\Phi}^j_{x_{l,m+s}} = \left(\widehat{\Phi}^j_{l+1,m+s} - \widehat{\Phi}^j_{l-1,m+s}\right)/2h, \tag{31.6}$$

$$\widehat{\Phi}^j_{x_{l\pm\frac{1}{2},m}} = \pm\left(\widehat{\Phi}^j_{l\pm1,m} - \widehat{\Phi}^j_{l,m}\right)/h, \tag{31.7}$$

$$\widehat{\Phi}^j_{x_{l,m\pm\frac{1}{2}}} = \left(\widehat{\Phi}^j_{l+1,m\pm1} - \widehat{\Phi}^j_{l-1,m\pm1} + \widehat{\Phi}^j_{l+1,m} - \widehat{\Phi}^j_{l-1,m}\right)/4h, \tag{31.8}$$

$$\widehat{\Phi}^j_{y_{l,m}} = \left(\widehat{\Phi}^j_{l,m+1} - \widehat{\Phi}^j_{l,m-1}\right)/2h, \tag{31.9}$$

$$\widehat{\Phi}^j_{y_{l,m\pm\frac{1}{2}}} = \pm\left(\widehat{\Phi}^j_{l,m\pm1} - \widehat{\Phi}^j_{l,m}\right)/h, \tag{31.10}$$

$$\widehat{\Phi}^j_{y_{l\pm\frac{1}{2},m}} = \left(\widehat{\Phi}^j_{l\pm1,m+1} - \widehat{\Phi}^j_{l\pm1,m-1} + \widehat{\Phi}^j_{l,m+1} - \widehat{\Phi}^j_{l,m-1}\right)/4h, \tag{31.11}$$

$$\widehat{\Phi}^j_{xx_{l,m+s}} = \left(\widehat{\Phi}^j_{l+1,m+s} - 2\widehat{\Phi}^j_{l,m+s} + \widehat{\Phi}^j_{l-1,m+s}\right)/h^2, \tag{31.12}$$

$$\widehat{\Phi}^j_{yy_{l+r,m}} = \left(\widehat{\Phi}^j_{l+r,m+1} - 2\widehat{\Phi}^j_{l+r,m} + \widehat{\Phi}^j_{l+r,m-1}\right)/h^2. \tag{31.13}$$

Analogous approximations are defined for $\Psi$, $\Psi_x$, $\Psi_y$ at the central point $(x_l, y_m, t_j)$ and the neighboring half-step points $(x_{l\pm\frac{1}{2}}, y_m, t_j)$ and $(x_l, y_{m\pm\frac{1}{2}}, t_j)$. Using the above approximations, we define

$$\begin{aligned}\widehat{G}^j_{l\pm\frac{1}{2},m} &= g\left(x_{l\pm\frac{1}{2}}, y_m, t_j, \widehat{\Phi}^j_{l\pm\frac{1}{2},m}, \widehat{\Phi}^j_{t_{l\pm\frac{1}{2},m}} \widehat{\Phi}^j_{x_{l\pm\frac{1}{2},m}}, \widehat{\Phi}^j_{y_{l\pm\frac{1}{2},m}}, \widehat{\Psi}^j_{l\pm\frac{1}{2},m}, \widehat{\Psi}^j_{x_{l\pm\frac{1}{2},m}}, \widehat{\Psi}^j_{y_{l\pm\frac{1}{2},m}}\right)\\ &= G^j_{l\pm\frac{1}{2},m} + \frac{k^2}{6}S_1 + \frac{h^2}{24}S_2 \pm O(k^2h + h^3),\end{aligned} \tag{32.1}$$

$$\begin{aligned}\widehat{G}^j_{l,m\pm\frac{1}{2}} &= g\left(x_l, y_{m\pm\frac{1}{2}}, t_j, \widehat{\Phi}^j_{l,m\pm\frac{1}{2}}, \widehat{\Phi}^j_{t_{l,m\pm\frac{1}{2}}}, \widehat{\Phi}^j_{x_{l,m\pm\frac{1}{2}}}, \widehat{\Phi}^j_{y_{l,m\pm\frac{1}{2}}}, \widehat{\Psi}^j_{l,m\pm\frac{1}{2}}, \widehat{\Psi}^j_{x_{l,m\pm\frac{1}{2}}}, \widehat{\Psi}^j_{y_{l,m\pm\frac{1}{2}}}\right)\\ &= G^j_{l,m\pm\frac{1}{2}} + \frac{k^2}{6}S_1 + \frac{h^2}{24}S_3 \pm O(k^2h + h^3),\end{aligned} \tag{32.2}$$

where

$$S_1 = 6\tau\Bigg[\Phi_{002}\left(\frac{\partial g}{\partial \Phi}\right)^j_{l,m} + \Psi_{002}\left(\frac{\partial g}{\partial \Psi}\right)^j_{l,m} + \Phi_{102}\left(\frac{\partial g}{\partial \Phi_x}\right)^j_{l,m} + \Psi_{102}\left(\frac{\partial g}{\partial \Psi_x}\right)^j_{l,m} + \Phi_{012}\left(\frac{\partial g}{\partial \Phi_y}\right)^j_{l,m}$$
$$+\Psi_{012}\left(\frac{\partial g}{\partial \Psi_y}\right)^j_{l,m} + \Phi_{003}\left(\frac{\partial g}{\partial \Phi_t}\right)^j_{l,m}\Bigg],$$

$$S_2 = \Bigg[3\Phi_{200}\left(\frac{\partial g}{\partial \Phi}\right)^j_{l,m} + 3\Psi_{200}\left(\frac{\partial g}{\partial \Psi}\right)^j_{l,m} + 3\Phi_{201}\left(\frac{\partial g}{\partial \Phi_t}\right)^j_{l,m} + \Phi_{300}\left(\frac{\partial g}{\partial \Phi_x}\right)^j_{l,m} + \Psi_{300}\left(\frac{\partial g}{\partial \Psi_x}\right)^j_{l,m}$$
$$+(3\Phi_{210}+4\Phi_{030})\left(\frac{\partial g}{\partial \Phi_y}\right)^j_{l,m} + (3\Psi_{210}+4\Psi_{030})\left(\frac{\partial g}{\partial \Psi_y}\right)^j_{l,m}\Bigg],$$

$$S_3 = \Bigg[3\Phi_{020}\left(\frac{\partial g}{\partial \Phi}\right)^j_{l,m} + 3\Psi_{020}\left(\frac{\partial g}{\partial \Psi}\right)^j_{l,m} + 3\Phi_{021}\left(\frac{\partial g}{\partial \Phi_t}\right)^j_{l,m} + \Phi_{030}\left(\frac{\partial g}{\partial \Phi_y}\right)^j_{l,m} + \Psi_{030}\left(\frac{\partial g}{\partial \Psi_y}\right)^j_{l,m}$$
$$+(3\Phi_{120}+4\Phi_{300})\left(\frac{\partial g}{\partial \Phi_x}\right)^j_{l,m} + (3\Psi_{120}+4\Psi_{300})\left(\frac{\partial g}{\partial \Psi_x}\right)^j_{l,m}\Bigg].$$

In order to achieve high order accuracy, we require following additional approximations:

$$\hat{\hat{\Phi}}^j_{l,m} = \hat{\Phi}^j_{l,m} + h^2\left(p_1\hat{\Phi}^j_{xx_{l,m}} + p_2\hat{\Phi}^j_{yy_{l,m}}\right), \tag{33.1}$$

$$\hat{\hat{\Psi}}^j_{l,m} = \hat{\Psi}^j_{l,m} + h^2\left(q_1\hat{\Psi}^j_{xx_{l,m}} + q_2\hat{\Psi}^j_{yy_{l,m}}\right), \tag{33.2}$$

$$\hat{\hat{\Phi}}^j_{t_{l,m}} = \hat{\Phi}^j_{t_{l,m}} + p_3\left(\hat{\Phi}^j_{t_{l+1,m}} - 2\hat{\Phi}^j_{t_{l,m}} + \hat{\Phi}^j_{t_{l-1,m}}\right) + p_4\left(\hat{\Phi}^j_{t_{l,m+1}} - 2\hat{\Phi}^j_{t_{l,m}} + \hat{\Phi}^j_{t_{l,m-1}}\right), \tag{33.3}$$

$$\hat{\hat{\Phi}}^j_{x_{l,m}} = \hat{\Phi}^j_{x_{l,m}} + h\left[p_5\left(\hat{\Psi}^j_{l+1,m} - \hat{\Psi}^j_{l-1,m}\right) + p_6\left(\hat{\Phi}^j_{yy_{l+1,m}} - \hat{\Phi}^j_{yy_{l-1,m}}\right)\right], \tag{33.4}$$

$$\hat{\hat{\Phi}}^j_{y_{l,m}} = \hat{\Phi}^j_{y_{l,m}} + h\left[p_7\left(\hat{\Psi}^j_{l,m+1} - \hat{\Psi}^j_{l,m-1}\right) + p_8\left(\hat{\Phi}^j_{xx_{l,m+1}} - \hat{\Phi}^j_{xx_{l,m-1}}\right)\right], \tag{33.5}$$

$$\hat{\hat{\Psi}}^j_{x_{l,m}} = \hat{\Psi}^j_{x_{l,m}} + h\left[q_3\left(\hat{G}^j_{l+\frac{1}{2},m} - \hat{G}^j_{l-\frac{1}{2},m}\right) + q_4\left(\hat{\Phi}^j_{tt_{l+1,m}} - \hat{\Phi}^j_{tt_{l-1,m}}\right) + q_5\left(\hat{\Psi}^j_{yy_{l+1,m}} - \hat{\Psi}^j_{yy_{l-1,m}}\right)\right] + h^2\left(q_6\hat{\Psi}^j_{xx_{l,m}} + q_7\hat{\Psi}^j_{yy_{l,m}}\right), \tag{33.6}$$

$$\hat{\hat{\Psi}}^j_{y_{l,m}} = \hat{\Psi}^j_{y_{l,m}} + h\left[q_8\left(\hat{G}^j_{l,m+\frac{1}{2}} - \hat{G}^j_{l,m-\frac{1}{2}}\right) + q_9\left(\hat{\Phi}^j_{tt_{l,m+1}} - \hat{\Phi}^j_{tt_{l,m-1}}\right) + q_{10}\left(\hat{\Psi}^j_{xx_{l,m+1}} - \hat{\Psi}^j_{xx_{l,m-1}}\right)\right] + h^2\left(q_{11}\hat{\Psi}^j_{xx_{l,m}} + q_{12}\hat{\Psi}^j_{yy_{l,m}}\right), \tag{33.7}$$

where $p_i,\ i = 1, \ldots, 8$ and $q_j,\ j = 1, \ldots, 12$ are the parameters to be suitably determined. By using the approximations (31.1)–(31.13) and (32.1) and (32.2), from (33.1) to (33.7), we obtain

$$\hat{\hat{\Phi}}^j_{l,m} = \Phi^j_{l,m} + \tau k^2\Phi_{002} + \frac{h^2}{6}S_4 + O(k^2h^2 + h^4), \tag{34.1}$$

$$\hat{\hat{\Psi}}^j_{l,m} = \Psi^j_{l,m} + \tau k^2\Psi_{002} + \frac{h^2}{6}S_5 + O(k^2h^2 + h^4), \tag{34.2}$$

$$\hat{\hat{\Phi}}^j_{t_{l,m}} = \Phi^j_{t_{l,m}} + \frac{k^2}{6}\Phi_{003} + \frac{h^2}{6}S_6 + O(k^2h + h^4), \tag{34.3}$$

$$\hat{\hat{\Phi}}^j_{x_{l,m}} = \Phi^j_{x_{l,m}} + \tau k^2\Phi_{102} + \frac{h^2}{6}S_7 + O(k^2h^2 + h^4), \tag{34.4}$$

$$\hat{\hat{\Phi}}^j_{y_{l,m}} = \Phi^j_{y_{l,m}} + \tau k^2\Phi_{012} + \frac{h^2}{6}S_8 + O(k^2h^2 + h^4), \tag{34.5}$$

$$\widehat{\widehat{\Psi}}_{x_{l,m}}^{j} = \Psi_{x_{l,m}}^{j} + \tau k^2 \Psi_{102} + \frac{h^2}{6} S_9 + O(k^2 h^2 + h^4), \tag{34.6}$$

$$\widehat{\widehat{\Psi}}_{y_{l,m}}^{j} = \Psi_{y_{l,m}}^{j} + \tau k^2 \Psi_{012} + \frac{h^2}{6} S_{10} + O(k^2 h^2 + h^4), \tag{34.7}$$

where

$$\begin{aligned}
S_4 &= 6(p_1 \Phi_{200} + p_2 \Phi_{020}),\\
S_5 &= 6(q_1 \Psi_{200} + q_2 \Psi_{020}),\\
S_6 &= 6(p_3 \Phi_{201} + p_4 \Phi_{021}),\\
S_7 &= (1 + 12 p_5) \Phi_{300} + 12(p_5 + p_6) \Phi_{120},\\
S_8 &= (1 + 12 p_7) \Phi_{030} + 12(p_7 + p_8) \Phi_{210},\\
S_9 &= (1 + 6 q_3 A_{000}) \Psi_{300} + 6(q_3 A_{000} + 2 q_5) \Psi_{120} + 6(q_3 A_{100} + q_6) \Psi_{200} + 6(q_3 A_{100} + q_7) \Psi_{020} + 6(q_3 + 2 q_4) \Psi_{102},\\
S_{10} &= (1 + 6 q_8 A_{000}) \Psi_{030} + 6(q_8 A_{000} + 2 q_{10}) \Psi_{210} + 6(q_8 A_{010} + q_{11}) \Psi_{200} + 6(q_8 A_{010} + q_{12}) \Psi_{020} + 6(q_8 + 2 q_9) \Psi_{012}.
\end{aligned}$$

Next, we define

$$\widehat{\widehat{G}}_{l,m}^{j} = g\big(x_l, y_m, t_j, \widehat{\widehat{\Phi}}_{l,m}^{j}, \widehat{\widehat{\Phi}}_{t_{l,m}}^{j} \widehat{\widehat{\Phi}}_{x_{l,m}}^{j}, \widehat{\widehat{\Phi}}_{y_{l,m}}^{j}, \widehat{\widehat{\Psi}}_{l,m}^{j}, \widehat{\widehat{\Psi}}_{x_{l,m}}^{j}, \widehat{\widehat{\Psi}}_{y_{l,m}}^{j}\big). \tag{35}$$

With the help of Taylor series and the approximations (34.1)–(34.7), from (35), we get

$$\widehat{\widehat{G}}_{l,m}^{j} = G_{l,m}^{j} + \frac{k^2}{6} S_1 + \frac{h^2}{6} S_{11} + O(k^2 h^2 + h^4), \tag{36}$$

where

$$S_{11} = S_4 \left(\frac{\partial g}{\partial \Phi}\right)_{l,m}^{j} + S_5 \left(\frac{\partial g}{\partial \Psi}\right)_{l,m}^{j} + S_6 \left(\frac{\partial g}{\partial \Phi_t}\right)_{l,m}^{j} + S_7 \left(\frac{\partial g}{\partial \Phi_x}\right)_{l,m}^{j} + S_8 \left(\frac{\partial g}{\partial \Phi_y}\right)_{l,m}^{j} + S_9 \left(\frac{\partial g}{\partial \Psi_x}\right)_{l,m}^{j} + S_{10} \left(\frac{\partial g}{\partial \Psi_y}\right)_{l,m}^{j}$$

Then at each grid point $(x_l, y_m, t_j)$, $l, m = 1, \ldots, N$, $j = 1, 2, \ldots,$ an approximation with accuracy $O(k^2 + h^4)$ for the solution of differential equations (30.1)-(30.2) may be written as

$$\left[\delta_x^2 + \delta_y^2 + \frac{1}{6}\delta_x^2 \delta_y^2\right] \widehat{\Phi}_{l,m}^{j} = \frac{h^2}{12}\left[\widehat{\Psi}_{l+1,m}^{j} + \widehat{\Psi}_{l-1,m}^{j} + \widehat{\Psi}_{l,m+1}^{j} + \widehat{\Psi}_{l,m-1}^{j} + 8 \widehat{\Psi}_{l,m}^{j}\right] + O(k^2 h^4 + h^6), \tag{37.1}$$

$$\begin{aligned}
&\left[L_1(\delta_x^2 + \delta_y^2) + L_2 \delta_x^2 \delta_y^2\right] \widehat{\Psi}_{l,m}^{j} + \frac{h^2}{12}\left[P_1 \widehat{\Phi}_{tt_{l+1,m}}^{j} + P_2 \widehat{\Phi}_{tt_{l-1,m}}^{j} + P_3 \widehat{\Phi}_{tt_{l,m+1}}^{j} + P_4 \widehat{\Phi}_{tt_{l,m-1}}^{j} + 8 \widehat{\Phi}_{tt_{l,m}}^{j}\right]\\
&\quad = \frac{h^2}{3}\left[K_1 \widehat{G}_{l+\frac{1}{2},m}^{j} + K_2 \widehat{G}_{l-\frac{1}{2},m}^{j} + K_3 \widehat{G}_{l,m+\frac{1}{2}}^{j} + K_4 \widehat{G}_{l,m-\frac{1}{2}}^{j} - \widehat{\widehat{G}}_{l,m}^{j}\right] + \widehat{T}_{l,m}^{j},
\end{aligned} \tag{37.2}$$

where

$$L_1 = B_{000} + \frac{h^2}{12}(B_{200} + B_{020}) - \frac{h^2}{6 B_{000}}\left(B_{100}^2 + B_{010}^2\right), \quad L_2 = \frac{1}{6} B_{000},$$

$$P_1 = \left(1 - \frac{h B_{100}}{B_{000}}\right),\ P_2 = \left(1 + \frac{h B_{100}}{B_{000}}\right),\ P_3 = \left(1 - \frac{h B_{010}}{B_{000}}\right),\ P_4 = \left(1 + \frac{h B_{010}}{B_{000}}\right),$$

$$K_1 = \left(1 - \frac{h B_{100}}{2 B_{000}}\right),\ K_2 = \left(1 + \frac{h B_{100}}{2 B_{000}}\right),\ K_3 = \left(1 - \frac{h B_{010}}{2 B_{000}}\right),\ K_4 = \left(1 + \frac{h B_{010}}{2 B_{000}}\right),$$

and

$$\widehat{T}_{l,m}^{j} = \frac{h^4}{36}[(S_2 + S_3) - 2 S_{11}] + k^2 h^2 \left[S_1 - A_{000}\, \tau(\Psi_{202} + \Psi_{022}) - \frac{1}{12}\Phi_{004}\right] + O(k^2 h^4 + h^6) \tag{38}$$

In order to obtain the difference method of $O(k^2 + h^4)$, the coefficient of $h^4$ in (38) must be zero, which gives the values of parameters

$$p_1 = p_2 = p_3 = p_4 = \frac{1}{4}, \quad p_5 = p_7 = \frac{1}{8}, \quad p_6 = p_8 = 0,$$

$$q_1 = q_2 = \frac{1}{4}, \quad q_3 = q_8 = \frac{1}{4 A_{000}}, \quad q_4 = q_9 = -\frac{1}{8 A_{000}}, \quad q_5 = q_{10} = 0, \quad q_6 = q_7 = -\frac{A_{100}}{4 A_{000}}, \quad q_{11} = q_{12} = -\frac{A_{010}}{4 A_{000}}.$$

Thus we obtain a difference method of $O(k^2 + h^4)$ for the differential equations (30.1)-(30.1) and the local truncation error reduces to $\widehat{T}_{l,m}^{j} = O(k^2 h^2 + h^6)$, for arbitrary $\tau$.

Since the schemes (37.1) and (37.2) are three-level method in time, to start the computation we need to know $\phi^1_{l,m}$ and $\psi^1_{l,m}$ of desired accuracy at the first time level, $t = k$. Since $\phi$ and $\phi_t$ are known at $t = 0$, this implies all their successive tangential derivatives are known at $t = 0$, i.e. the values

$$\frac{\partial^{r+s}\phi^0_{l,m}}{\partial x^r \partial y^s}, \quad \frac{\partial^{r+s+1}\phi^0_{l,m}}{\partial x^r \partial y^s \partial t}, \quad r, s = 0, 1, \ldots$$

are known which are sufficient to determine $\psi \equiv \phi_{xx} + \phi_{yy}$ and $\nabla^4\phi = \phi_{xxxx} + 2\phi_{xxyy} + \phi_{yyyy}$ initially. Approximations for $\phi$ and $\psi$ of $O(k^2)$ at $t = k$ may be written as

$$\phi^1_{l,m} = \phi^0_{l,m} + k\phi^0_{t_{l,m}} + \frac{k^2}{2}\phi^0_{tt_{l,m}} + O(k^3), \tag{39}$$

$$\psi^1_{l,m} = \psi^0_{l,m} + k\psi^0_{t_{l,m}} + \frac{k^2}{2}\psi^0_{tt_{l,m}} + O(k^3). \tag{40}$$

From (6), we have

$$\frac{\partial^2\phi}{\partial t^2} = -B(x, y, t)\nabla^4\phi + g(x, y, t, \phi, \phi_t, \phi_x, \phi_y, \nabla^2\phi, \nabla^2\phi_x, \nabla^2\phi_y). \tag{41}$$

Differentiating (41) twice successively with respect to $x$ and $y$ and then adding, we obtain

$$\frac{\partial^2\psi}{\partial t^2} = \nabla^2[-B(x, y, t)\nabla^4\phi + g(x, y, t, \phi, \phi_t, \phi_x, \phi_y, \nabla^2\phi, \nabla^2\phi_x, \nabla^2\phi_y)]. \tag{42}$$

Thus using the initial values and their successive tangential partial derivative values, from (41) and (42), we can evaluate the values of $\phi^0_{tt_l}$ and $\psi^0_{tt_l}$, respectively, and then finally, from (39) and (40) we can compute the values of $\phi$ and $\psi$ at the first time level, i.e. at $t = k$.

For the quasi-linear case, when $B = B(x, y, t, \phi, \nabla^2\phi)$, we make the following central difference approximations in the formulas (37.1) and (37.2):

$$B_{100} = \left(\widehat{B}^j_{l+1,m} - \widehat{B}^j_{l-1,m}\right)/(2h) + O(h^2), \tag{43.1}$$

$$B_{010} = \left(\widehat{B}^j_{l,m+1} - \widehat{B}^j_{l,m-1}\right)/(2h) + O(h^2), \tag{43.2}$$

$$B_{200} = \left(\widehat{B}^j_{l+1,m} - 2\widehat{B}^j_{l,m} + \widehat{B}^j_{l-1,m}\right)/h^2 + O(h^2), \tag{43.3}$$

$$B_{020} = \left(\widehat{B}^j_{l,m+1} - 2\widehat{B}^j_{l,m} + \widehat{B}^j_{l,m-1}\right)/h^2 + O(h^2), \tag{43.4}$$

for

$$\widehat{B}^j_{l,m} = B\left(x_l, y_m, t_j, \widehat{\Phi}^j_{l,m}, \widehat{\Psi}^j_{l,m}\right),$$
$$\widehat{B}^j_{l\pm1,m} = B\left(x_{l\pm1}, y_m, t_j, \widehat{\Phi}^j_{l\pm1,m}, \widehat{\Psi}^j_{l\pm1,m}\right),\ \widehat{B}^j_{l,m\pm1} = B\left(x_l, y_{m\pm1}, t_j, \widehat{\Phi}^j_{l,m\pm1}, \widehat{\Psi}^j_{l,m\pm1}\right).$$

Using the central differences (43.1)–(43.4) in the difference schemes (37.1) and (37.2), the order of the scheme is retained and hence the $O(k^2 + h^4)$ difference method for the two-dimensional fourth order quasi-linear parabolic equation is attained.

## 4. Operator splitting scheme

Consider the linear two-dimensional time dependent fourth order PDE (10) including mixed terms with a time derivative of the Laplacian. Let us introduce a new variable defined by

$$\omega(x, y, t) = \nu\nabla^2\phi - \frac{\partial\phi}{\partial t}.$$

Since $\phi_0(x, y)$ is assumed to be smooth enough, the initial condition for $\omega(x, y, t)$ can be derived by successively differentiating $\phi_0(x, y)$ partially with respect to '$x$' and '$y$' so that $\nabla^2\phi(x, y, 0) = \frac{\partial^2\phi_0}{\partial x^2} + \frac{\partial^2\phi_0}{\partial y^2}$. Likewise using the initial conditions (2), (7) and the boundary conditions (3), $\frac{\partial\phi}{\partial t}$ and $\nabla^2\phi$ are determined on all sides of the solution domain $\Omega \times (0, T]$. Hence $\phi$ and $\omega$ are known initially and on the boundary of the solution region.

Eq. (10) can be written as coupled system of two diffusion equations as

$$\nu\nabla^2\phi - \frac{\partial\phi}{\partial t} = \omega(x, y, t), \quad (x, y, t) \in \Omega \times (0, T], \tag{44.1}$$

$$\nu\nabla^2\omega - \frac{\partial\omega}{\partial t} = d(x,y,t), \quad (x,y,t) \in \Omega \times (0,T]. \tag{44.2}$$

subject to the initial conditions

$$\phi(x,y,0) = \phi_0(x,y), \ \ \omega(x,y,0) = \frac{\partial^2\phi_0}{\partial x^2} + \frac{\partial^2\phi_0}{\partial y^2} - \phi_1(x,y), \ \ (x,y) \in \Omega \tag{45}$$

and the boundary conditions

$$\phi = h_1(x,y,t), \ \ \omega = h_4(x,y,t), \ \ (x,y,t) \in \partial\Omega \times (0,T] \tag{46}$$

An $O(k^2+h^4)$ two-level implicit difference scheme for Eqs. (44.1) and (44.2) may be written as

$$\begin{aligned}&\left[1 + \frac{1}{12}(1-6\lambda\nu)\delta_x^2 + \frac{1}{12}(1-6\lambda\nu)\delta_y^2 - \frac{\lambda\nu}{12}\delta_x^2\delta_y^2\right]\phi_{l,m}^{j+1}\\ &= \left[1 + \frac{1}{12}(1+6\lambda\nu)\delta_x^2 + \frac{1}{12}(1+6\lambda\nu)\delta_y^2 + \frac{\lambda\nu}{12}\delta_x^2\delta_y^2\right]\phi_{l,m}^{j} - \frac{k}{2}\left[1+\frac{1}{12}\left(\delta_x^2+\delta_y^2\right)\right]\left(\omega_{l,m}^{j+1}+\omega_{l,m}^{j}\right),\end{aligned} \tag{47.1}$$

$$\begin{aligned}&\left[1 + \frac{1}{12}(1-6\lambda\nu)\delta_x^2 + \frac{1}{12}(1-6\lambda\nu)\delta_y^2 - \frac{\lambda\nu}{12}\delta_x^2\delta_y^2\right]\omega_{l,m}^{j+1}\\ &= \left[1 + \frac{1}{12}(1+6\lambda\nu)\delta_x^2 + \frac{1}{12}(1+6\lambda\nu)\delta_y^2 + \frac{\lambda\nu}{12}\delta_x^2\delta_y^2\right]\omega_{l,m}^{j} - \frac{k}{3}\sum d\end{aligned} \tag{47.2}$$

where

$$\sum d = \overline{d}_{l+\frac{1}{2},m}^{j} + \overline{d}_{l-\frac{1}{2},m}^{j} + \overline{d}_{l,m+\frac{1}{2}}^{j} + \overline{d}_{l,m-\frac{1}{2}}^{j} - \overline{d}_{l,m}^{j},$$

for

$$\overline{d}_{l,m}^{j} = d\left(x_l, y_m, t_j + \frac{k}{2}\right), \ \overline{d}_{l\pm\frac{1}{2},m}^{j} = d\left(x_{l\pm\frac{1}{2}}, y_m, t_j + \frac{k}{2}\right), \ \overline{d}_{l,m\pm\frac{1}{2}}^{j} = d\left(x_l, y_{m\pm\frac{1}{2}}, t_j + \frac{k}{2}\right).$$

The linear difference equations (47.1) and (47.2) requires solution of a system of equations with a large band width at each time level. Also, it is difficult to study the stability region for the schemes (47.1) and (47.2). In this regard, we add certain higher order terms in such a way that the accuracy of the scheme remains unaltered. The considered schemes (47.1) and (47.2) may be factorized as

$$[X_1][Y_1]\phi_{l,m}^{j+1} = [X_2][Y_2]\phi_{l,m}^{j} - \frac{k}{2}\left[1+\frac{1}{12}\left(\delta_x^2+\delta_y^2\right)\right]\left(\omega_{l,m}^{j+1}+\omega_{l,m}^{j}\right) \equiv [R_\phi] \tag{48.1}$$

$$[X_1][Y_1]\omega_{l,m}^{j+1} = [X_2][Y_2]\omega_{l,m}^{j} - \frac{k}{3}\sum d \equiv [R_\omega], \ \ l,m = 1,\ldots,N, \ \ j = 0,1,\ldots, \tag{48.2}$$

for

$$\begin{aligned}X_1 &= 1 + \frac{1}{12}(1-6\lambda\nu)\delta_x^2, \ \ Y_1 = 1 + \frac{1}{12}(1-6\lambda\nu)\delta_y^2,\\ X_2 &= 1 + \frac{1}{12}(1+6\lambda\nu)\delta_x^2, \ \ Y_2 = 1 + \frac{1}{12}(1+6\lambda\nu)\delta_y^2.\end{aligned}$$

The additional terms added in (48.1) and (48.2) are of high orders and do not affect the accuracy of the scheme.

In order to facilitate the computation, we may rewrite (48.1)-(48.2) in two-step operator split form (see [1,6,11–13]) as

$$[Y_1]\phi_{l,m}^{*\,j+1} = [R_\phi], \tag{49.1}$$

$$[X_1]\phi_{l,m}^{j+1} = \phi_{l,m}^{*\,j+1}, \tag{49.2}$$

$$[Y_1]\omega_{l,m}^{*\,j+1} = [R_\omega], \tag{49.3}$$

$$[X_1]\omega_{l,m}^{j+1} = \omega_{l,m}^{*\,j+1}, \quad l,m = 1,\ldots,N, \ \ j = 0,1,\ldots, \tag{49.4}$$

where $\phi_{l,m}^{*}$ and $\omega_{l,m}^{*}$ are the intermediate values between $j$th and $(j+1)$th time levels. The intermediate boundary values required for solving (49.1) and (49.3) can be obtained from (49.2) and (49.4), respectively. The left-hand sides of (49.1)–(49.3) and (49.2)–(49.4) are factorizations into $y$- and $x$-differences, respectively, which permits us to solve by sweeping first in the $y$-direction and then in the $x$-direction. These sweeps involve only the solution of tri-diagonal systems and hence easily solvable.

For stability, we use the Von-Neumann method. At each grid point $(x_l, y_m, t_j)$, we assume $\phi^j_{l,m} = \xi^j e^{i\beta l} e^{i\gamma m}$ and $\omega^j_{l,m} = \eta^j e^{i\beta l} e^{i\gamma m}$, for $i = \sqrt{-1}$, $\xi$, $\eta$ being the amplification factors, $\beta$, $\gamma$ being the phase angles. Using these in the homogenous part of the schemes (48.1) and (48.2), we obtain the following matrix vector form as

$$\boldsymbol{z}^{j+1} = \boldsymbol{M}\boldsymbol{z}^j, \tag{50}$$

where

$$\boldsymbol{z} = [\boldsymbol{\phi}\ \ \boldsymbol{\omega}]^t, \quad \boldsymbol{M} = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}^{-1} \begin{bmatrix} c & -b \\ 0 & c \end{bmatrix},$$

for

$$a = \left[1 - \frac{1}{3}(1 - 6\lambda\nu)\sin^2\frac{\beta}{2}\right]\left[1 - \frac{1}{3}(1 - 6\lambda\nu)\sin^2\frac{\gamma}{2}\right],\ b = \frac{k}{2}\left(1 - \frac{1}{3}\sin^2\frac{\beta}{2} - \frac{1}{3}\sin^2\frac{\gamma}{2}\right),$$
$$c = \left[1 - \frac{1}{3}(1 + 6\lambda\nu)\sin^2\frac{\beta}{2}\right]\left[1 - \frac{1}{3}(1 + 6\lambda\nu)\sin^2\frac{\gamma}{2}\right].$$

The eigenvalues of the matrix $\boldsymbol{M}$ are given by

$$\xi_{1,2} = \frac{c}{a} = \frac{\left[1 - \frac{1}{3}(1 + 6\lambda\nu)\sin^2\frac{\beta}{2}\right]\left[1 - \frac{1}{3}(1 + 6\lambda\nu)\sin^2\frac{\gamma}{2}\right]}{\left[1 - \frac{1}{3}(1 - 6\lambda\nu)\sin^2\frac{\beta}{2}\right]\left[1 - \frac{1}{3}(1 - 6\lambda\nu)\sin^2\frac{\gamma}{2}\right]} \tag{51}$$

It follows that the eigenvalues of the amplification matrix $\boldsymbol{M}$ are less than equal to unity in modulus for all phase angles $\beta, \gamma \in [-\pi, \pi]$ and hence the operator splitting schemes (49.1)–(49.4) is unconditionally stable.

## 5. Numerical illustrations

Using the approximations (13.1)–(13.14) with $\theta = \frac{1}{2}$ into the differential equations (11.1) and (11.2), we get the two-level central difference scheme of accuracy $O(k^2 + h^2)$:

$$\left(\delta_x^2 + \delta_y^2\right)\overline{\phi}^j_{l,m} = h^2\overline{\psi}^j_{l,m} + O(k^2h^2 + h^4), \tag{52.1}$$

$$\overline{A}^j_{l,m}\left(\delta_x^2 + \delta_y^2\right)\overline{\psi}^j_{l,m} + h^2\overline{\phi}^j_{t_{l,m}} = h^2\overline{F}^j_{l,m} + O(k^2h^2 + h^4), \quad l, m = 1, \ldots, N;\ j = 0, 1, \ldots, \tag{52.2}$$

A three-level implicit cental difference method of $O(k^2 + h^2)$ for the solution of differential equations (30.1) and (30.2) may be expressed as:

$$\left(\delta_x^2 + \delta_y^2\right)\widehat{\phi}^j_{l,m} = h^2\widehat{\psi}^j_{l,m} + O(k^2h^2 + h^4), \tag{53.1}$$

$$B_{000}\left(\delta_x^2 + \delta_y^2\right)\widehat{\psi}^j_{l,m} + h^2\widehat{\phi}^j_{tt_{l,m}} = h^2\widehat{G}^j_{l,m} + O(k^2h^2 + h^4), \quad l, m = 1, \ldots, N,\ j = 1, 2, \ldots, \tag{53.2}$$

In this section, we have solved some fourth order PDEs of physical importance using the proposed schemes for validating the theoretical deductions with numerical values in connection with order of magnitude and accuracy in solution values. We choose $\tau = 0.5$ for the computation of problems of type (6). The right side homogenous function, the initial and boundary conditions may be obtained from the analytical solution as a trial process. The coupled system of linear difference equations were solved by Gauss–Seidel iterative method while the Newton–Raphson method has been used for nonlinear system of equations (see [16,17]). The initial vector (0,0) was used in all cases and the iterative procedure was terminated when the absolute error tolerance $|u^{(k+1)} - u^{(k)}| + |v^{(k+1)} - v^{(k)}| \le 2 \times 10^{-12}$ was achieved. The fourth order linear equation (10) has been solved using the operator splitting schemes (49.1)–(49.4).

**Example 1.** (*Nonlinear extended Fisher–Kolmogorov equation,* [18])

We consider the EFK Eq. (4) with the initial condition

$$\phi(x, y, 0) = x^3(1 - x)^3 y^3(1 - y)^3, \quad (x, y) \in \Omega \tag{54}$$

and the boundary conditions

$$\phi = 0, \quad \nabla^2\phi = 0, \quad (x, y, t) \in \partial\Omega \times (0, T]. \tag{55}$$

The profile of the numerical solution of Eq. (4) for $\gamma = 0.1$ subject to (54) and (55) is shown in Fig. 1. This solution is computed by using the schemes (17.1) and (17.2) with $k = 0.025$ and $h = 0.025$ at $T = 1.0$. The graph shows the same behavior as in [18].

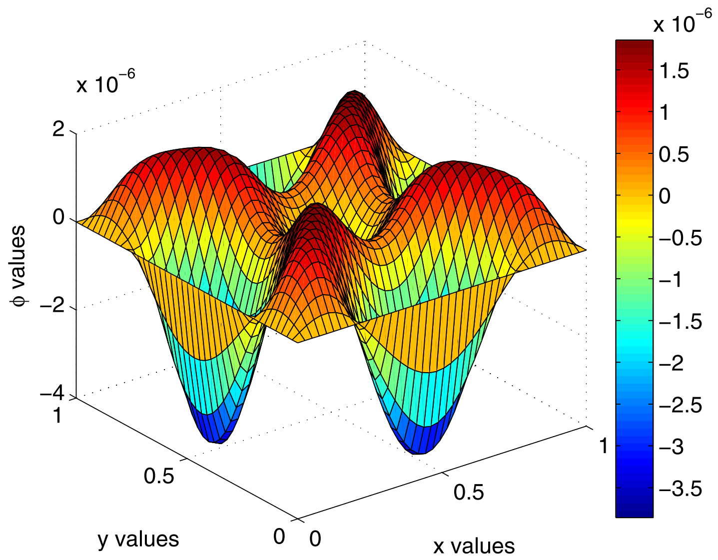



**Fig. 1.** The Profile of numerical solution for Example 1 with $\gamma = 0.1$, $J = 40$ and $N+1 = 40$ at $T = 1.0$ by the difference schemes (17.1) and (17.2).

**Table 1**
Maximum absolute errors and order of convergence for Example 2 at $T = 1.0$ for different values of $k = \frac{1}{J}$ and $h = \frac{1}{N+1}$ with $\gamma = 0.01$.

| | | | Khiari and Omrani [18] | | Present method (17.1) and (17.2) | |
|---|---|---|---|---|---|---|
| $J$ | $N+1$ | | MAE | Order | MAE | Order |
| 15 | 15 | $\phi$ | 0.0064 | – | 3.6811(-04) | – |
| | | $\nabla^2\phi$ | | | 2.0267(-02) | – |
| 20 | 20 | $\phi$ | 0.0038 | – | 1.6715(-04) | – |
| | | $\nabla^2\phi$ | | | 1.4472(-02) | – |
| 25 | 25 | $\phi$ | 0.0022 | – | 9.4791(-05) | – |
| | | $\nabla^2\phi$ | | | 6.3222(-03) | – |
| 30 | 30 | $\phi$ | 0.0016 | 2.001 | 6.0840(-05) | 2.5970 |
| | | $\nabla^2\phi$ | | | 5.0584(-03) | 2.0024 |
| 40 | 40 | $\phi$ | 8.1310(-04) | 2.2245 | 3.2096(-05) | 2.3807 |
| | | $\nabla^2\phi$ | | | 2.6170(-03) | 2.4673 |
| 50 | 50 | $\phi$ | 4.4541(-04) | 2.3043 | 1.9722(-05) | 2.2649 |
| | | $\nabla^2\phi$ | | | 1.5913(-03) | 1.9902 |

**Example 2.** (*Nonlinear nonhomogenous extended Fisher–Kolmogorov equation,* [18])

We consider the nonhomogeneous EFK equation

$$\frac{\partial \phi}{\partial t} + \gamma \nabla^4 \phi - \nabla^2 \phi + \phi^3 - \phi = f(x, y, t), \quad (x, y) \in \Omega, \quad t > 0 \tag{56}$$

with the initial condition

$$\phi(x, y, 0) = \sin(2\pi x)\sin(2\pi y), \quad (x, y) \in \Omega \tag{57}$$

subject to the boundary conditions

$$\phi = 0, \quad \nabla^2 \phi = 0, \quad (x, y, t) \in \partial\Omega \times (0, 1]. \tag{58}$$

where $f(x, y, t) = \left(\sin(2\pi x)^2 \sin(2\pi y)^2 e^{-t^2} - 2 + 64\gamma\pi^4 + 8\pi^2\right)\sin(2\pi x)\sin(2\pi y)e^{-t}$. The exact solution is

$$\phi(x, y, t) = \sin(2\pi x)\sin(2\pi y)e^{-t}$$

The maximum absolute errors (MAEs) and order of convergence of the schemes (17.1) and (17.2) for $\gamma = 0.01$ with $k = h$ is presented in Table 1 for different sizes of the grid at $T = 1.0$ along with the existing result in literature [18]. It is observed that with this choice of time step size, the convergence estimated numerically is equal to two since the theoretical order of convergence of the $O(k^2 + kh^2 + h^4)$ schemes (17.1) and (17.2) becomes $O(h^2)$ for $k = h$. The obtained numerical solutions are competent with the available results in literature and the proposed method provides relatively better solutions. Table 1 shows that for the same parameters as in [18], the present methods (17.1) and (17.2) not only gave quite satisfactory results with respect to the exact solutions but are also much accurate than the solutions obtained by the difference scheme

**Table 2**
Maximum absolute errors and order of convergence for difference methods (17.1) and (17.2) to Example 2 for different values of $\gamma$ at $T = 1.0$ with $\lambda = (k/h^2) = 1.6$.

| Mesh | | $\gamma = 0.1$ | | $\gamma = 0.01$ | | $\gamma = 0.001$ | |
|---|---|---|---|---|---|---|---|
| | | MAE | Order | MAE | Order | MAE | Order |
| 9 × 9 | $\phi$ | 2.3012(−03) | – | 2.1319(−03) | – | 1.9909(−03) | – |
| | $\nabla^2\phi$ | 2.3213(−01) | – | 2.1866(−01) | – | 2.0867(−01) | – |
| 17 × 17 | $\phi$ | 1.4007(−04) | 4.0382 | 1.2897(−04) | 4.0470 | 1.1968(−04) | 4.0562 |
| | $\nabla^2\phi$ | 1.4326(−02) | 4.0182 | 1.3457(−02) | 4.0223 | 1.2747(−02) | 4.0330 |
| 33 × 33 | $\phi$ | 8.6923(−06) | 4.0103 | 7.9924(−06) | 4.0123 | 7.4061(−06) | 4.0143 |
| | $\nabla^2\phi$ | 8.9196(−04) | 4.0055 | 8.3705(−04) | 4.0069 | 7.9202(−04) | 4.0085 |

**Table 3**
Maximum absolute errors and order of convergence for Example 3 at $T = 1.0$.

| Mesh | | $O(k^2 + kh^2 + h^4)$-method | | $O(k^2 + h^2)$-method | |
|---|---|---|---|---|---|
| | | (17.1)–(17.2) | | (52.1)–(52.2) | |
| | | MAE | Order | MAE | Order |
| 9 × 9 | $\phi$ | 1.0215(−06) | – | 1.2604(−04) | – |
| | $\nabla^2\phi$ | 2.0668(−05) | – | 1.9112(−03) | – |
| 17 × 17 | $\phi$ | 6.3728(−08) | 4.0026 | 3.2122(−05) | 1.9722 |
| | $\nabla^2\phi$ | 1.3122(−06) | 3.9773 | 6.3442(−04) | 1.5910 |
| 33 × 33 | $\phi$ | 4.0226(−09) | 3.9857 | 8.0365(−06) | 1.9989 |
| | $\nabla^2\phi$ | 8.0887(−08) | 4.0199 | 1.8003(−04) | 1.8172 |

in [18]. Also, the maximum absolute errors are reported in Table 2 at $T = 1.0$ for a fixed mesh ratio parameter $\lambda = 1.6$ for various values of $\gamma$ and it is shown that the order of convergence of the proposed methods (17.1) and (17.2) is indeed four for fixed mesh ratio parameter.

**Example 3.** (*Nonlinear nonhomogenous Sivashinsky equation,* [24])
Consider the nonhomogenous Sivashinsky equation

$$\frac{\partial \phi}{\partial t} + \nabla^4\phi - \nabla^2\left(\frac{1}{2}\phi^2 - 2\phi\right) + 2\phi = f(x, y, t), \quad (x, y) \in \Omega, \quad t > 0, \tag{59}$$

where

$$f(x, y, t) = 2\cos(x + y + t) - \sin(x + y + t) - 2\sin^2(x + y + t) + 2\cos^2(x + y + t),$$

for which the exact solution is

$$\phi(x, y, t) = \cos(x + y + t)$$

In Table 3, the numerical errors are obtained for $\lambda = 1.6$ with different grid spacing so as to depict order of magnitude of the proposed schemes (17.1) and (17.2) at $T = 1.0$. The maximum absolute errors are plotted in Fig. 2 on a log–log scale.

**Example 4.** (*Two-dimensional unsteady singular biharmonic equation,* [23])
Consider the two-dimensional fourth order problem with singular coefficient as

$$\nabla^4\phi + \frac{\partial \phi}{\partial t} + \frac{1}{x}\phi_x = f(x, y, t), \quad 0 < x, y < 1, \quad t > 0, \tag{60}$$

where $f(x, y, t)$ may be obtained from the analytical solution

$$\phi(x, y, t) = e^{-t}x^2\sin(\pi y).$$

We have solved the above example at $T = 1.0$ for various grid spacing and obtained maximum absolute errors and order of magnitude in Table 4. It is evident that both order and accuracy are preserved even in the vicinity of singularity $x = 0$ using the half-step discretization (17.1) and (17.2).

**Example 5.** (*Quasi-linear problem,* [21,23])
Consider the following quasi-linear problem:

$$(1 + x^2 + y^2 + \phi^2)\nabla^4\phi + \frac{\partial^2\phi}{\partial t^2} = f(x, y, t), \quad 0 < x, y < 1, \quad t > 0, \tag{61}$$

where $f(x, y, t) = [4\pi^4(1 + x^2 + y^2 + \sinh^2(t)\sin^2(\pi x)\sin^2(\pi y)) + 1]\sinh(t)\sin(\pi x)\sin(\pi y)$. The exact solution is

$$\phi(x, y, t) = \sinh(t)\sin(\pi x)\sin(\pi y).$$

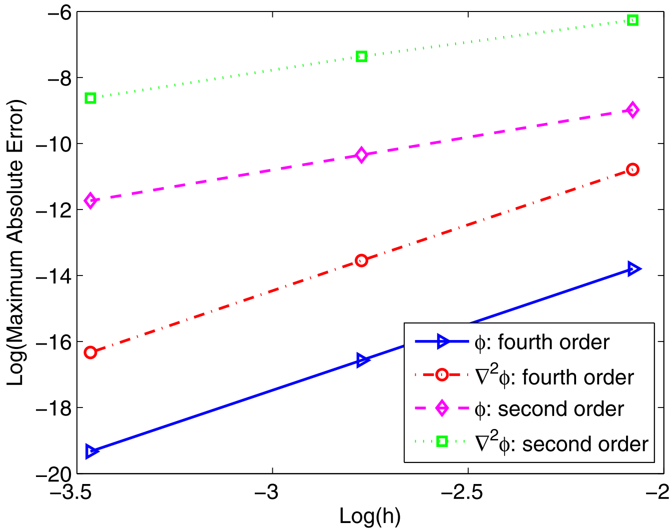



**Fig. 2.** Order of convergence for the nonhomogenous Sivashinsky equation (59), Example 3 at $T = 1.0$ by the difference schemes (17.1)–(17.2) and (52.1)–(52.2).

**Table 4**
Maximum absolute errors and order of convergence for Example 4 at $T = 1.0$.

| Mesh | | $O(k^2 + kh^2 + h^4)$-method | | $O(k^2 + h^2)$-method | |
|---|---|---|---|---|---|
| | | (17.1)–(17.2) | | (52.1)–(52.2) | |
| | | MAE | Order | MAE | Order |
| $9 \times 9$ | $\phi$ | 2.1630(−05) | – | 1.1964(−03) | – |
| | $\nabla^2\phi$ | 1.0212(−03) | – | 8.0216(−02) | – |
| $17 \times 17$ | $\phi$ | 1.2775(−06) | 4.0816 | 2.8932(−04) | 2.0480 |
| | $\nabla^2\phi$ | 7.5769(−05) | 3.7525 | 2.6963(−02) | 1.5729 |
| $33 \times 33$ | $\phi$ | 8.0135(−08) | 3.9947 | 7.2426(−05) | 1.9981 |
| | $\nabla^2\phi$ | 5.1260(−06) | 3.8857 | 7.5793(−03) | 1.8308 |

**Table 5**
Maximum absolute errors and order of convergence for Example 1 at $T = 1.0$.

| Mesh | | $O(k^2 + kh^2 + h^4)$-method | | $O(k^2 + h^2)$-method | |
|---|---|---|---|---|---|
| | | (37.1)–(37.2) | | (53.1)–(53.2) | |
| | | MAE | Order | MAE | Order |
| $9 \times 9$ | $\phi$ | 8.3258(−04) | – | 3.1325(−02) | – |
| | $\nabla^2\phi$ | 1.2414(−02) | – | 3.1292(−01) | – |
| $17 \times 17$ | $\phi$ | 4.9068(−05) | 4.0847 | 7.4100(−03) | 2.0798 |
| | $\nabla^2\phi$ | 7.3726(−04) | 4.0737 | 7.1479(−02) | 2.1302 |
| $33 \times 33$ | $\phi$ | 3.0541(−06) | 4.0060 | 1.8433(−03) | 2.0072 |
| | $\nabla^2\phi$ | 4.5897(−05) | 4.0057 | 1.7765(−02) | 2.0085 |

The maximum absolute errors in the solution variable $\phi$, the moment $\nabla^2\phi$ and order of convergence computed by difference methods (37.1)–(37.2) and (53.1)–(53.2) are listed in Table 5 for $\lambda = 1.6$ at $T = 1.0$.

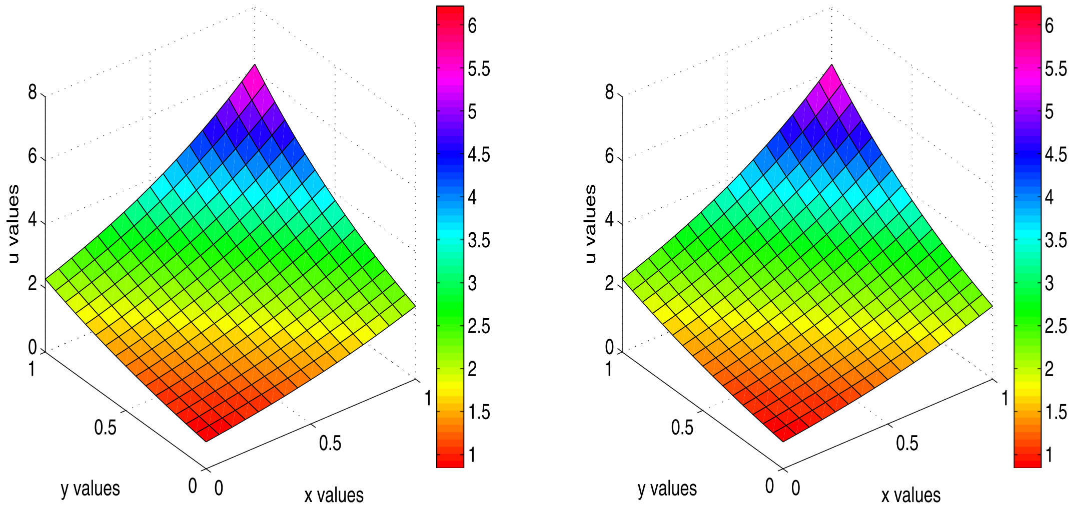
**Example 6.** (*Nonlinear nonhomogenous Boussinesq equation,* [3,29])
Consider the nonhomogenous Boussinesq equation of the form:

$$\nabla^4\phi + \frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi - \nabla^2\phi^2 = g(x, y, t), \quad 0 < x, y < 1, \quad t > 0 \tag{62}$$

The exact solution is given by $\phi(x, y, t) = e^{x+y}\sin t$. The homogeneous function $g(x, y, t)$ and the initial, the boundary conditions can be obtained using the exact solution as a test procedure. The maximum absolute errors for $\phi$ and $\nabla^2\phi$ are given in Table 6. The 3D graphs of numerical solution by (37.1) and (37.2) vs exact solution are plotted in Fig. 3 at $T = 1.0$ for $h = 1/16$ and $\lambda = 1.6$.

**Table 6**
Maximum absolute errors and order of convergence for Example 6 at $T = 1.0$.

| Mesh | | $O(k^2 + kh^2 + h^4)$-method (37.1)–(37.2) MAE | Order | $O(k^2 + h^2)$-method (53.1)–(53.2) MAE | Order |
|---|---|---|---|---|---|
| $9 \times 9$ | $\phi$ | 9.0079(−05) | – | 7.0599(−04) | – |
| | $\nabla^2\phi$ | 1.8056(−03) | – | 5.0558(−03) | – |
| $17 \times 17$ | $\phi$ | 5.6818(−06) | 3.9868 | 2.0274(−04) | 1.8000 |
| | $\nabla^2\phi$ | 1.1199(−04) | 4.0110 | 1.6523(−03) | 1.6135 |
| $33 \times 33$ | $\phi$ | 3.5493(−07) | 4.0007 | 5.1966(−05) | 1.9640 |
| | $\nabla^2\phi$ | 7.0397(−06) | 3.9917 | 4.3685(−04) | 1.9193 |




(a) Numerical Solution (b) Exact Solution

**Fig. 3.** Plots of numerical and exact solution for the nonhomogenous Boussinesq equation (62), Example 6 at $T = 1.0$ for $0 \le x \le 1$, $0 \le y \le 1$ with $h = 1/16$ by (37.1) and (37.2).

**Table 7**
Maximum absolute errors and order of convergence for Example 7 at $T = 1.0$.

| Mesh | | $O(k^2 + kh^2 + h^4)$-method (37.1)–(37.2) MAE | Order | $O(k^2 + h^2)$-method (53.1)–(53.2) MAE | Order |
|---|---|---|---|---|---|
| $9 \times 9$ | $\phi$ | 9.0043(−04) | – | 3.3940(−02) | – |
| | $\nabla^2\phi$ | 1.6216(−02) | – | 3.6299(−01) | – |
| $17 \times 17$ | $\phi$ | 5.4690(−05) | 4.0413 | 8.2498(−03) | 2.0406 |
| | $\nabla^2\phi$ | 9.8162(−04) | 4.0461 | 8.7453(−02) | 2.0534 |
| $33 \times 33$ | $\phi$ | 3.4076(−06) | 4.0044 | 2.0533(−03) | 2.0064 |
| | $\nabla^2\phi$ | 6.1135(−05) | 4.0051 | 2.1762(−02) | 2.0067 |

**Example 7.** (*Fourth order singular parabolic equation,* [23])
Consider the two dimensional fourth order singular problem which is second order with respect to time:

$$\nabla^4\phi + \frac{\partial \phi^2}{\partial t^2} + \frac{1}{y}\nabla^2\phi = g(x, y, t), \quad 0 < x, y < 1, \quad t > 0 \tag{63}$$

where $g(x, y, t)$ may be obtained from the considered analytical solution

$$\phi(x, y, t) = \sinh(t)\sin(\pi x)\sin(\pi y).$$

The numerical results for this problem are shown in Table 7 for various mesh sizes at $T = 1.0$ with $\lambda = 1.6$. These results demonstrate the accuracy of methods (37.1) and (37.2) since the errors reduce with increasing grid spacing.

**Example 8.** [23] We solve the two dimensional fourth order linear parabolic problem (10) with the proposed spatially fourth order operator splitting schemes (49.1)–(49.4) for various values of $\nu$ at $T = 1.0$ with $\lambda = 1.6$ for different mesh sizes. The

**Table 8**
Maximum absolute errors and order of convergence for the operator splitting method (49.1)–(49.4) to Example 8 for different values of $\nu$ at $T = 1.0$ with $\lambda = (k/h^2) = 1.6$.

| Mesh | | $\nu = 0.1$ | | $\nu = 0.01$ | | $\nu = 0.001$ | |
|---|---|---|---|---|---|---|---|
| | | MAE | Order | MAE | Order | MAE | Order |
| $9 \times 9$ | $\phi$ | 4.3002(−05) | – | 7.9389(−05) | – | 1.0823(−04) | – |
| | $\nabla^2\phi$ | 2.0237(−05) | – | 1.9213(−05) | – | 2.3410(−05) | – |
| $17 \times 17$ | $\phi$ | 2.6935(−06) | 3.9968 | 4.9445(−06) | 4.0050 | 6.1058(−06) | 4.1478 |
| | $\nabla^2\phi$ | 1.2793(−06) | 3.9836 | 1.2068(−06) | 3.9928 | 1.3585(−06) | 4.1070 |
| $33 \times 33$ | $\phi$ | 1.6932(−07) | 3.9917 | 3.0897(−07) | 4.0003 | 3.8776(−07) | 3.9769 |
| | $\nabla^2\phi$ | 7.9456(−08) | 4.0000 | 7.5670(−08) | 3.9953 | 8.2036(−08) | 4.0496 |
| $65 \times 65$ | $\phi$ | 1.0583(−08) | 3.9999 | 1.9310(−08) | 4.0000 | 2.4203(−08) | 4.0019 |
| | $\nabla^2\phi$ | 5.0047(−09) | 3.9979 | 4.7339(−09) | 3.9986 | 5.1572(−09) | 3.9916 |

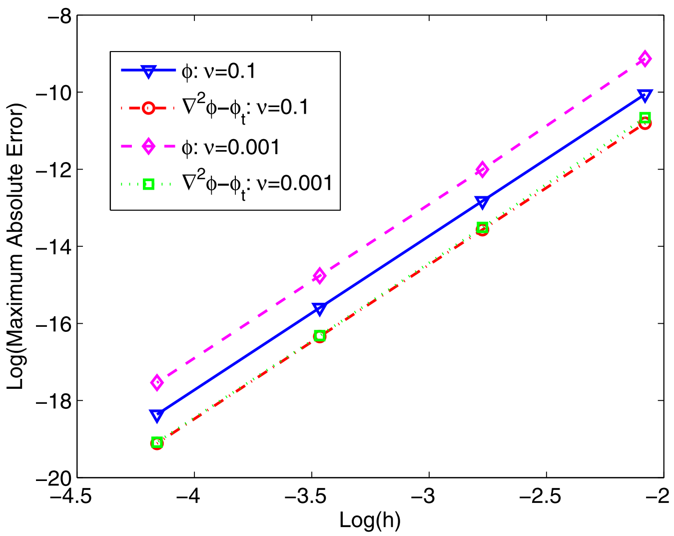



**Fig. 4.** Order of convergence of the operator splitting schemes (49.1) and (49.4) for Example 8 at $T = 1.0$ for different values of $\nu$.

numerical errors for $\phi$, its time-dependent Laplacian $\left(\nabla^2 - \frac{\partial}{\partial t}\right)\phi(x, y, t)$ and order of convergence are reported in Table 8. In Fig. 4, we display in a Log/Log scale the errors (shown numerically in Table 8) for $\nu = 0.1, 0.001$ at $T = 1.0$. It is seen from Fig. 4 that the slope of the graph is almost constant around four.

## 6. Concluding remarks

We have described 9-point compact finite difference schemes to solve time dependent PDEs with fourth order spatial derivatives in two-space dimensions. The given fourth order differential equation is reduced into a system of two second-order equations subject to Dirichlet type boundary conditions. These schemes have second order accuracy in time and fourth order accuracy in space. The foremost advantage of developing these new methods which uses half-step points of the grid is that these are directly applicable to singular problems without requiring any fictitious points or particular treatment at the point of singularity. Eventually, computational complexity has been reduced and for fixed value of the mesh ratio parameter $\lambda = k/h^2$, the derived methods perform like fourth order methods which is evident from the numerical results. The simulations presented in this work exhibit a good agreement with regard to the analytical solutions. The efficacy of the proposed approach has been tested on several physical problems including the good Boussinesq equation, Sivashinsky equation and the EFK equation. The proposed two-level methods (17.1) and (17.2) give more accurate results for the EFK equation than those discussed in [18]. Using the fourth order accurate methods, we are able to simulate the dynamics of two-space dimensional fourth order time-dependent problems with sparser grids and lesser time steps in comparison with the standard second order schemes. Another utility of the methods is that these can be applied with ease to solve nonlinear partial differential equations without linearizing the nonlinear term. The present approach is combined with the Newton's iterative method to solve nonlinear problems. We also discuss operator splitting method for solving a class of fourth order linear parabolic equation which allows multiple use of the one-dimensional tri-diagonal solver and is shown to be unconditionally stable. In addition, the solution of Laplacian which is often of interest in applied problems is also computed. Several numerical experiments are performed to demonstrate the performance of the proposed algorithms. It is hoped that schemes presented in the paper can be extended to facilitate numerical studies of higher order nonlinear parabolic problems arising from mathematical models of a diverse range of physical processes.

## References

[1] C. Andrade, S. McKee, High accuracy A.D.I. methods for fourth order parabolic equations with variable coefficients, J. Comput. Appl. Math. 3 (1977) 11–14.
[2] D.G. Aronson, H.F. Weinberger, Multidimensional nonlinear diffusion arising in population genetics, Adv. Math. 30 (1978) 33–76.
[3] M.J. Boussinesq, Theorie des ondes et des remous qui se propagent le long dun canal rectangular horizontal, J. Math. Pures Appl. 17 (1872) 55–108.
[4] M.M. Chawla, P.N. Shivakumar, An efficient finite difference method for two-point boundary value problems, Neural Parallel Sci. Comput. 4 (1996) 387–396.
[5] S.D. Conte, Numerical solution of vibration problems in two space variables, Pac. J. Math. 7 (1957) 1535–1544.
[6] S.D. Conte, R.T. Dames, On an alternating direction method for solving the plate problem with mixed boundary conditions, J. Assoc. Comput. Mach. 7 (1960) 264–273.
[7] P. Danumjaya, A.K. Pani, Finite Element Methods for the Extended Fisher–Kolmogorov Equation, in: Research Report: IMG-RR-2002-3, Industrial Mathematics Group, Department of Mathematics, IIT, Bombay.
[8] P. Danumjaya, A.K. Pani, Numerical methods for the extended Fisher–Kolmogorov (EFK) equation, Int. J. Numer. Anal. Model. 3 (2006) 186–210.
[9] G.T. Dee, W. Van Saarloos, Bistable systems with propagating fronts leading to pattern formation, Phys. Rev. Lett. 60 (1988) 2641–2644.
[10] D. He, On the $l^\infty$-norm convergence of a three-level linearly implicit finite difference method for the extended fisher kolmogorov equation in both 1d and 2d, Comput. Math. Appl. 71 (2016) 2594–2607.
[11] G. Fairweather, A.R. Mitchell, A high accuracy alternating direction method for the wave equation, IMA J. Appl. Math 1 (1965) 309–316.
[12] G. Fairweather, A.R. Gourlay, Some stable difference approximations to a fourth-order parabolic partial differential equation, Math. Comput. 21 (1967) 1–11.
[13] G. Fairweather, A.R. Mitchell, A new computational procedure for A.D.I. methods, SIAM J. Numer. Anal. 4 (1967) 163–170.
[14] G. Fairweather, Galerkin methods for vibration problems in two space variables, SIAM J. Numer. Anal. 9 (1972) 702–714.
[15] G. Gua, S. Lü, Unconditional stability of alternating difference schemes with intrinsic parallelism for two-dimensional fourth-order diffusion equation, Comput. Math. Appl. 71 (2016) 1944–1959.
[16] L.A. Hageman, D.M. Young, Applied Iterative Methods, Dover Publication, New York, 2004.
[17] C.T. Kelly, Iterative Methods for Linear and Non-linear Equations, SIAM Publications, Philadelphia, 1995.
[18] N. Khiari, K. Omrani, Finite difference discretization of the extended Fisher–Kolmogorov equation in two dimensions, Comput. Math. Appl. 62 (2011) 4151–4160.
[19] R.K. Mohanty, W. Dai, F. Han, A new high accuracy method for two-dimensional biharmonic equation with nonlinear third derivative terms: application to Navier–Stokes equations of motion, Int. J. Comput. Math. 92 (2015) 1574–1590.
[20] R.K. Mohanty, D. Kaur, High accuracy implicit variable mesh methods for numerical study of special types of fourth order non-linear parabolic equations, Appl. Math. Comput. 273 (2016) 678–696.
[21] R.K. Mohanty, D. Kaur, A class of quasi-variable mesh methods based on off-step discretization for the numerical solution of fourth-order quasi-linear parabolic partial differential equations, Adv. Differ. Equ. (2016) Paper No. 326, 29 pp.
[22] R.K. Mohanty, D. Kaur, Numerov type variable mesh approximations for 1d unsteady quasi-linear biharmonic problem: application to Kuramoto–Sivashinsky equation, Numer. Algorithm 74 (2017) 427–459.
[23] R.K. Mohanty, D. Kaur, High accuracy compact operator methods for two dimensional fourth order nonlinear parabolic partial differential equations, Comput. Methods Appl. Math. 17 (2017) 617–641.
[24] K. Omrani, Numerical methods and error analysis for the nonlinear Sivashinsky equation, Appl. Math. Comput. 189 (2007) 949–962.
[25] M. Rouis, K. Omrani, On the numerical solution of two dimensional model of an alloy solidification problem, Model. Numer. Simul. Mater. Sci. 6 (2016) 1–9.
[26] V.K. Saul'ev, A new difference method for the numerical solution of fourth-order parabolic equations, USSR Comput. Maths Math. Phys. 26 (1986) 147–153.
[27] S.P. Timoshenko, S. Woinowsky-Krieger, Theory of Plates and Shells, 2nd ed., McGraw-Hill, New York, 1959.
[28] E.H. Twizell, A.Q.M. Khaliq, A difference scheme with high accuracy in time for fourth-order parabolic equations, Comput. Methods Appl. Mech. Eng. 41 (1983) 91–104.
[29] G.B. Whitham, Linear and Nonlinear Waves, Wiley-Interscience, New York, 1974.
[30] T.P. Witelski, M. Bowen, ADI schemes for higher-order nonlinear diffusion equations, Appl. Numer. Math. 45 (2003) 331–351.