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Perfect codes in poset block spaces

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There is a limited class of perfect codes with respect to the classical Hamming metric. There are other kind of metrics with respect to which perfect codes have been investigated viz. poset metric, block metric and poset block metric. Given the minimal elements of a poset, a necessary and sufficient condition for 1-perfectness of a poset block code has been derived. A necessary and sufficient condition for a poset block code to be r-perfect has also been considered. Further, for each r, $1 \leq r \leq n-k$, a sufficient condition that ensures the existence of a poset block structure which turns a given code into an r-perfect poset block code has been obtained. Several illustrations of well known codes to be r-perfect for specific values of r have been explored.

Keywords: Poset block metric; perfect codes; weight enumerator.

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1. Introduction

It is well known that coding theory deals mainly with the study of linear subspaces of $\mathbb{F}_q^n$, the space of all n-tuples over $\mathbb{F}_q$, endowed with a metric, usually the Hamming metric. Codes in the Hamming space have been studied extensively in search for the optimal ones with respect to different parameters such as dimension of code, minimum distance, etc. Perfect codes have been of utmost interest due to their ideal

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structure and wide applicability. However, perfect codes are rare in the Hamming space.

Several attempts have been made to generalize the classical problems in coding theory by introducing a new metric on $\mathbb{F}_q^n$. In 1995, Brualdi *et al.* [3] introduced a metric derived from a partial order of coordinates of codewords, called the poset metric. Increasing the relations on a poset enlarges the distances and hence shrinks the metric balls. Codes such as the extended binary Hamming codes and extended Golay codes turn out to be perfect in the case of poset metric due to increased packing radius in this case. Poset metric has been investigated by several authors [1, 5, 8, 9].

Another generalization of the classical Hamming metric, called the block metric or π metric, was proposed by Feng *et al.* [6] by partitioning the set of all coordinate positions into a family of blocks. As opposed to the case of poset metric, enlarging the blocks diminishes distances and hence augments metric balls. Thus the packing radius of a code with respect to a block metric is smaller than its Hamming packing radius.

Firer *et al.* [2] introduced poset block metric by invoking partial order amongst the blocks. This metric has also been studied by Panek *et al.* [10]. The study of perfectness of codes in poset block spaces becomes all the more interesting due to opposite effects of poset and block metric on distances and hence on the packing radius of the code. The wide applications of perfect codes in covering problems, combinatorial configurations, group theory and diophantine number theory are essentially a motivating factor to search for perfectness with respect to poset block metric.

The paper has been organized as follows: In Sec. 2, basic definitions and concepts have been presented. In Sec. 3, a necessary and sufficient condition for a poset block code to be 1-perfect has been derived. The general case of an r-perfect poset block code has also been studied. A sufficient condition for a code to be an r-perfect poset block code in terms of the usual Hamming weight enumerator of the code has also been obtained.

Throughout the paper, index H has been used to denote the Hamming metric d_H, the parameters of a linear code in Hamming metric $(n, k, d_H)_H$, and the support $\mathrm{supp}_H(x) = \{i : x_i \neq 0\}$ of a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n$.

2. Poset Block Metric

In this section, basic definitions of poset block metric space have been provided.

Let $P = ([s], \leq)$ be a *poset* where $\leq$ is a partial order on $[s] = \{1, 2, \ldots, s\}$. An *ideal* of the poset P is a non-empty subset $I \subseteq [s]$ such that $i \in I$ and $j \leq i$ implies $j \in I$. Given a subset $X \subset [s]$, denote by $\langle X \rangle$ the smallest ideal containing X, called the *ideal generated by* X; if $X = i$ then we denote the ideal generated by X as $\langle i \rangle$ as in [2]. The *i-th level set* $\Gamma^{(t)}(P)$ of P is defined as

$$\Gamma^{(t)}(P) = \{i \in P : |\langle i \rangle| = t\}.$$

In particular, the elements of $\Gamma^{(1)}(P)$ are referred to as minimal elements of P.

Let $\pi : [s] \to \mathbb{N}$ be a map such that $n = \sum_{i=1}^{s} \pi(i)$. The map π is said to be a *labeling* of the poset P and the pair (P, π) is called a *poset block structure* over $[s]$. Denote $\pi(i)$ by k_i and take V_i as the $\mathbb{F}_q$-vector space $V_i = \mathbb{F}_q^{k_i}$ for every $1 \leq i \leq s$. Define V as

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_s$$

which is isomorphic to $\mathbb{F}_q^n$. Each $x \in \mathbb{F}_q^n$ can be uniquely decomposed as $x = (x_1, x_2, \ldots, x_s)$ where $x_i \in \mathbb{F}_q^{k_i}, 1 \leq i \leq s$. The *block support* or π-*support* and (P, π)-*weight* of $x \in \mathbb{F}_q^n$ are defined, respectively, as

$$\operatorname{supp}_\pi(x) = \{i : 1 \leq i \leq s, x_i \neq 0\}$$

and

$$w_{(P,\pi)}(x) = |\langle \operatorname{supp}_\pi(x) \rangle|$$

where $|\cdot|$ denotes the cardinality of the given set. For $x, y \in V$,

$$d_{(P,\pi)}(x, y) = w_{(P,\pi)}(x - y)$$

defines a metric over $\mathbb{F}_q^n$ called the *poset block metric* or (P, π)-*metric*. The pair $(\mathbb{F}_q^n, d_{(P,\pi)})$ is said to be a *poset block space*. A k-dimensional $\mathbb{F}_q$-linear subspace C of V is said to be an $(n, k, d_{(P,\pi)})$ (P, π)-code, where $\mathbb{F}_q^n$ is equipped with the poset block metric $d_{(P,\pi)}(\cdot, \cdot)$ and

$$d_{(P,\pi)}(C) = \min\{w_{(P,\pi)}(c) : 0 \neq c \in C\}$$

is the (P, π)-minimum distance of C.

Remark 2.1 ([2]). The poset metric d_P introduced by Brualdi *et al.* [3] can be realized as a particular case of poset block metric when the label π satisfies $\pi(i) = 1$ for all $i \in [s]$. The block metric d_π introduced by Feng *et al.* [6] can also be seen as a particular case of poset block metric when P is taken to be an anti-chain, i.e. no two elements are related. When $\pi(i) = 1$ for all $i \in [s]$ and P is an anti-chain then the poset block metric reduces to the classical Hamming metric d_H.

For $x \in \mathbb{F}_q^n$ and a non-negative integer r, the (P, π)-*ball* with center x and radius r is defined as the set of all those vectors of V whose distance from x in poset block metric is less than or equal to r. For all $x \in \mathbb{F}_q^n$,

$$|B_{(P,\pi)}(x, r)| = 1 + \sum_{i=1}^{r} \sum_{j=1}^{i} \sum_{I \in \theta_j(i)} \prod_{m \in \max(I)} (q^{k_m} - 1) \prod_{\ell < m; m \in \max(I)} q^{k_\ell}, \quad (2.1)$$

where $\theta_j(i) = \{I \in P : I \text{ ideal}, |I| = i, |\max(I)| = j\}$, $\max(I)$ is the set of maximal elements in the ideal I. A code C is said to be an r-*perfect* with respect to poset block metric if the (P, π)-balls of radius r centered at the codewords of C are pairwise disjoint and their union is $\mathbb{F}_q^n$.

3. Perfect Codes in Poset Block Spaces

In this section, necessary and sufficient conditions for a code to be perfect with respect to poset block metric have been discussed. In [4], the authors have classified all poset block structures which turn the extended binary Golay code and extended ternary Golay code into 1-perfect poset block codes. Generalizing these results in the following theorem, we classify the poset block structure for a code C to be 1-perfect in terms of minimal elements of the poset.

Theorem 3.1. *Let (P, π) be a poset block structure with $\Gamma^{(1)}(P) = \{1, 2, \ldots, t\}$, $t \in \mathbb{Z}^+$ and C be an (n, k) code over $\mathbb{F}_q^n$. Then C is a 1-perfect poset block code if and only if the following two conditions hold*:

$$1 - t + \sum_{i=1}^{t} q^{k_i} = q^{n-k}; \tag{3.1}$$

$$c \notin V_i \oplus V_j \quad \text{for each } c(\neq 0) \in C \text{ and for each } 1 \leq i, j \leq t. \tag{3.2}$$

Proof. Assume that C is a 1-perfect poset block code. Then $|C| \cdot |B_{(P,\pi)}(0, 1)| = q^n$, which implies that $|B_{(P,\pi)}(0, 1)| = q^{n-k}$. Using (2.1), we have

$$1 - t + \sum_{i=1}^{t} q^{k_i} = q^{n-k}.$$

Now we prove (3.2). Let, if possible, there exists a $c(\neq 0) \in C$ such that $c \in V_i \oplus V_j$ for some $1 \leq i, j \leq t$. If $i = j$, $c \in B_{(P,\pi)}(0, 1)$ which contradicts that C is a 1-perfect code. If $i \neq j$, $c = c_i + c_j$ where $c_i \in V_i$ and $c_j \in V_j$. This implies that $c_i \in B_{(P,\pi)}(0, 1) \cap B_{(P,\pi)}(c, 1)$, which contradicts that C is a 1-perfect code.

Conversely, assume that (3.1) and (3.2) hold. In view of (3.2), it is evident that the (P, π)-balls of radius 1 centered at the codewords of C cover the whole space. Now we prove that the (P, π)-balls of radius 1 centered at codewords of C are disjoint, which is equivalent to

$$B_{(P,\pi)}(0, 1) \cap B_{(P,\pi)}(c, 1) = \phi \quad \text{for all } c(\neq 0) \in C.$$

Else, there exists $x \in \mathbb{F}_q^n$ such that $x \in B_{(P,\pi)}(0, 1) \cap B_{(P,\pi)}(c, 1)$. Now $x \in B_{(P,\pi)}(0, 1)$ implies that $x \in V_i$ for some $1 \leq i \leq t$. Also, $x \in B_{(P,\pi)}(c, 1)$ implies that $c - x \in V_j$ for some $1 \leq j \leq t$. Combining the two, we have $c \in V_i \oplus V_j$, which contradicts (3.2). Thus C is a 1-perfect poset block code. □

Remark 3.1. For a given q, we can easily find values of t as follows:

From (3.1), we have

$$\sum_{i=1}^{t} q^{k_i} - q^{n-k} = t - 1$$

which implies that

$$t \equiv 1 \pmod{q}. \tag{3.3}$$

This allows us to look for permissible values of t if q is given and vice-versa. Thus for a given q, it is clear that $t \in \{q, q+1, 2q+1, \ldots\}$.

Remark 3.2. For $t = 1, 2, 3$ and 4, we discuss (3.1) and (3.2) as follows:

Case 1. For $t = 1$, (3.1) reduces to

$$q^{k_1} = q^{n-k}. \tag{3.4}$$

Take $k_1 = n - k$ in (3.4). Since (3.4) holds for all q, therefore if $\Gamma^{(1)}(P) = \{1\}$ then a code C is 1-perfect if and only if $c \notin V_1$ for each $c(\neq 0) \in C$.

Case 2. For $t = 2$, (3.3) does not hold for any q. Thus, no code can be 1-perfect with respect to a poset block metric having two minimal elements in the poset.

Case 3. For $t = 3$, (3.3) holds only when $q = 2$. Thus (3.1) becomes

$$2^{k_1-1} + 2^{k_2-1} + 2^{k_3-1} = 2^{n-k-1} + 1. \tag{3.5}$$

If k_1, k_2 and k_3 are positive integers > 1 then left-hand side is a multiple of 2 whereas right-hand side is not. Therefore, at least one of k_1, k_2 and k_3 must be 1.

Case 3.1. When any one of k_1, k_2 and k_3 is 1.

Without loss of generality, take $k_3 = 1$. Then (3.5) reduces to

$$2^{k_1-1} + 2^{k_2-1} = 2^{n-k-1}.$$

If $k_1 \neq k_2$, assume that $k_1 > k_2$. The above equation becomes

$$2^{k_2-1}(2^{k_1-k_2} + 1) = 2^{n-k-1}.$$

This equation does not hold since 2 is the only prime occurring on right-hand side whereas this is not the case for left-hand side. Similarly $k_1 \not< k_2$. Therefore $k_1 = k_2$. Then we get

$$2^{k_1} = 2^{n-k-1}$$

which gives us $k_1 = n - k - 1$. Thus (3.5) holds when $k_1 = k_2 = n - k - 1$ and $k_3 = 1$.

Case 3.2. When any two of k_1, k_2 and k_3 are 1.

Without loss of generality, take $k_2 = k_3 = 1$. Then (3.5) reduces to

$$2^{k_1-1} + 1 = 2^{n-k-1}.$$

This equation does not hold.

Case 3.3. When each of k_1, k_2 and k_3 is 1. Then (3.5) reduces to

$$1 + 1 + 1 = 2^{n-k-1} + 1$$

which implies that $n-k=2$. Thus (3.5) holds when $k_1=k_2=k_3=1$ and $n-k=2$. Therefore, if $\Gamma^{(1)}(P)=\{1,2,3\}$ then a code C is 1-perfect if and only if $q=2$ and

$$k_1=k_2=n-k-1, \quad k_3=1 \quad \text{and} \quad c \notin V_i \oplus V_j \quad \text{for each } 1 \leq i,\ j \leq 3 \tag{3.6}$$

or

$$k_1=k_2=k_3=1, \quad n-k=2 \quad \text{and} \quad c \notin V_i \oplus V_j \quad \text{for each } 1 \leq i,\ j \leq 3. \tag{3.7}$$

Conditions in (3.6) are strong, however, (3.7) is satisfied for any binary MDS code with $n-k=2$. The only binary MDS code with $n-k=2$ is $(3,1,3)$ binary code (c.f. [7, Theorem 2.4.4]).

Case 4. For $t=4$, (3.3) holds only when $q=3$. Thus (3.1) becomes

$$3^{k_1-1}+3^{k_2-1}+3^{k_3-1}+3^{k_4-1}=3^{n-k-1}+1. \tag{3.8}$$

Proceeding as in Case 3, we get:

If $\Gamma^{(1)}(P)=\{1,2,3,4\}$ then a code C is 1-perfect if and only if $q=3$ and

$$k_1=k_2=k_3=n-k-1, \quad k_4=1 \quad \text{and}$$
$$c \notin V_i \oplus V_j \quad \text{for each } 1 \leq i,\ j \leq 4 \tag{3.9}$$

or

$$k_1=k_2=k_3=k_4=1, \quad n-k=2 \quad \text{and}$$
$$c \notin V_i \oplus V_j \quad \text{for each } 1 \leq i,\ j \leq 4. \tag{3.10}$$

Conditions in (3.9) are strong, however, (3.10) is satisfied for any ternary MDS code with $n-k=2$. The class of ternary MDS codes with $n-k=2$ is non-empty as ternary $(4,2)$ Hamming code with generator matrix

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

belongs to this class.

The following theorem is a generalization of [8, Proposition 1] and presents a necessary and sufficient condition for a code to be r-perfect in poset block space. It may be noted that characterization of an r-perfect code in terms of the blocks of a given poset block structure, i.e. generalization of Theorem 3.1 to r-perfect case does not seem possible. Assume that empty set in a partition is permitted.

Theorem 3.2. *Let (P,π) be a poset block structure and C be an (n,k) code over $\mathbb{F}_q$. Then C is an r-perfect (P,π)-code if and only if the following two conditions are satisfied*:

(1) (*The sphere packing condition*)
$|B_{(P,\pi)}(0,r)|=q^{n-k}$;

(2) (*The partition condition*)
For any nonzero codeword c and any partition $\{X, Y\}$ of $supp_\pi(c)$ (which corresponds to partition $\{c_X, c_Y\}$ of c), either $w_{(P,\pi)}(c_X) \geq r+1$ or $w_{(P,\pi)}(c_Y) \geq r+1$.

Proof. First assume that C is an r-perfect (P,π)-code. Then $|C| \cdot |B_{(P,\pi)}(0,r)| = q^n$, which implies that $|B_{(P,\pi)}(0,r)| = q^{n-k}$. Now we prove the partition condition. Assume, on the contrary, that there exists a nonzero codeword c and a partition $\{X, Y\}$ of $\text{supp}_\pi(c)$ such that $w_{(P,\pi)}(c_X) < r+1$ and $w_{(P,\pi)}(c_Y) < r+1$. This implies that $w_{(P,\pi)}(c_X) \leq r$ and $w_{(P,\pi)}(c - c_X) \leq r$, which further implies that $c_X \in B_{(P,\pi)}(0,r) \cap B_{(P,\pi)}(c,r)$. This contradicts that C is an r-perfect code.

Conversely, assume that both the sphere packing condition and partition condition hold. It is evident that (P,π)-balls of radius r centered at the codewords of C cover $\mathbb{F}_q^n$. Now we prove that these balls are disjoint. Suppose that

$$x \in B_{(P,\pi)}(0,r) \cap B_{(P,\pi)}(c,r).$$

This implies that $w_{(P,\pi)}(x) \leq r$ and $w_{(P,\pi)}(c-x) \leq r$. By definition of (P,π)-metric, we have $|\langle \text{supp}_\pi(x) \rangle| \leq r$ and $|\langle \text{supp}_\pi(c-x) \rangle| \leq r$. Since $\{X = \text{supp}_\pi(x) \cap \text{supp}_\pi(c), Y = \text{supp}_\pi(c) \backslash \text{supp}_\pi(x) \cap \text{supp}_\pi(c)\}$ is a partition of $\text{supp}_\pi(c)$, the partition condition implies that either $w_{(P,\pi)}(c_X) \geq r+1$ or $w_{(P,\pi)}(c_Y) \geq r+1$. Also, $w_{(P,\pi)}(c_X) \leq w_{(P,\pi)}(x) \leq r$ therefore $w_{(P,\pi)}(c_Y) \geq r+1$. Now

$$w_{(P,\pi)}(c-x) = |\langle \text{supp}_\pi(c-x) \rangle| \geq |\langle \text{supp}_\pi(c_Y) \rangle| \geq r+1$$

which is a contradiction since $|\langle \text{supp}_\pi(c-x) \rangle| \leq r < r+1$. Therefore $B_{(P,\pi)}(0,r) \cap B_{(P,\pi)}(c,r) = \phi$. □

Deduction 3.1. *The conditions for 1-perfectness as stated in Theorem* 3.1 *are equivalent to the conditions obtained in Theorem* 3.2 *by substituting $r = 1$ for the case when $\Gamma^{(1)}(P) = \{1, 2, \ldots, t\}$.*

Proof. It is straightforward that the sphere packing condition for the case under consideration is equivalent to (3.2). We prove that the partition condition is equivalent to (3.2). First, assume that the partition condition holds. Let, if possible, there exists a nonzero codeword c in C such that $c \in V_i \oplus V_j$ for some $1 \leq i, j \leq t$. This implies that there exists a partition $\{i, j\}$ of $\text{supp}_\pi(c)$ such that $w_{(P,\pi)}(c_i) = 1$ and $w_{(P,\pi)}(c_j) = 1$, which contradicts the partition condition.

Now assume that (3.2) holds. Let, if possible, there exists a nonzero codeword c and a partition $\{X, Y\}$ of $\text{supp}_\pi(c)$ such that $w_{(P,\pi)}(c_X) < 2$ and $w_{(P,\pi)}(c_Y) < 2$. The remaining proof has been bifurcated into three cases.

Case 1. When $w_{(P,\pi)}(c_X) = 0$ and $w_{(P,\pi)}(c_Y) = 0$.

This would imply that $c = 0$, which is a contradiction.

Case 2. When $w_{(P,\pi)}(c_X) = 1$ and $w_{(P,\pi)}(c_Y) = 0$.

This implies that $w_{(P,\pi)}(c) = w_{(P,\pi)}(c_X) = 1$ which in turn implies that $X = \{i\}$ where $1 \leq i \leq t$. Thus, we have $c \in V_i$ for some $i \in \Gamma^{(1)}(P)$, which is a contradiction. Similarly, the case when $w_{(P,\pi)}(c_X) = 0$ and $w_{(P,\pi)}(c_Y) = 1$ can be ruled out by interchanging the roles of X and Y.

Case 3. When $w_{(P,\pi)}(c_X) = 1$ and $w_{(P,\pi)}(c_Y) = 1$.

Then $c \in V_i \oplus V_j$ for some $1 \leq i, j \leq t$ with $i \neq j$, which again is a contradiction. Therefore the partition condition holds. □

Deduction 3.2. *For the case of NRT poset block metric, i.e. when P is a chain, the conditions for r-perfectness as obtained in Theorem* 3.2 *are equivalent to the condition in* [2, *Proposition* 3.1].

Proof. First we assume that both the sphere packing condition and partition condition hold. We show the existence of a linear transformation

$$L : V_{r+1} \oplus V_{r+2} \oplus \cdots \oplus V_s \to V_1 \oplus V_2 \oplus \cdots \oplus V_r$$

such that

$$C = \{(L(v), v) : v \in V_{r+1} \oplus V_{r+2} \oplus \cdots \oplus V_s\}.$$

Assume that $(u, v), (u', v) \in C$. Then $(u, v) - (u', v) = (u - u', 0) \in C$. Let $c = (u-u', 0)$. Consider a partition of $\text{supp}_\pi(c)$ of the form $\{X = \text{supp}_\pi(u-u'), Y = \phi\}$. Then $w_{(P,\pi)}(c_X) < r$ and $w_{(P,\pi)}(c_Y) < r$, which contradicts the partition condition. Therefore every element $v \in V_{r+1} \oplus V_{r+2} \oplus \cdots \oplus V_s$ determines a unique element $u \in V_1 \oplus V_2 \oplus \cdots \oplus V_r$ and hence determines a function, say L. Let $v, w \in V_{r+1} \oplus V_{r+2} \oplus \cdots \oplus V_s$ and $\alpha, \beta \in \mathbb{F}_q$. Then $(L(v), v) \in C$ and $(L(w), w) \in C$. Since C is a linear subspace, we have $\alpha(L(v), v) + \beta(L(w), w) \in C$ which implies that $(\alpha L(v) + \beta L(w), \alpha v + \beta w) \in C$. Therefore, $L(\alpha v + \beta w) = \alpha L(v) + \beta L(w)$. Thus we have a linear transformation L such that

$$C = \{(L(v), v) : v \in V_{r+1} \oplus V_{r+2} \oplus \cdots \oplus V_s\}.$$

Conversely, assume that such a linear transformation exists. This implies that

$$|V_{r+1} \oplus V_{r+2} \oplus \cdots \oplus V_s| = |C| = q^k.$$

Since P is a chain, we get

$$|B_{(P,\pi)}(0, r)| = |V_1 \oplus V_2 \oplus \cdots \oplus V_r| = q^{n-k}.$$

Let, if possible, there exists a nonzero codeword c and a partition $\{X, Y\}$ of $\text{supp}_\pi(c)$ such that $w_{(P,\pi)}(c_X) < r$ and $w_{(P,\pi)}(c_Y) < r$. Since P is a chain, we get $c_X, c_Y \in V_1 \oplus V_2 \oplus \cdots \oplus V_r$. Thus $c \in V_1 \oplus V_2 \oplus \cdots \oplus V_r$. This gives us $L(0) = c$, which is a contradiction since $L(0) = 0$. □

The following theorem provides a sufficient condition for a code to be an r-perfect poset block code in terms of the Hamming weight enumerator of the code.

Theorem 3.3. *Let C be an $(n,k,d)_H$ code over $\mathbb{F}_q$ with the Hamming weight enumerator $\sum_{i=0}^{n} A_i x^i$. If*

$$\binom{n}{n-k} - A_d\binom{n-d}{n-k-d} - A_{d+1}\binom{n-(d+1)}{n-k-(d+1)} - \cdots - A_{n-k}\binom{k}{0} > 0 \tag{3.11}$$

then for each r, $1 \le r \le n-k$, there exists a poset block structure (P,π) such that C is an r-perfect (P,π)-code.

Proof. First we show that there exists a subset of $[n]$ of cardinality $n-k$ that does not contain the support of any codeword of C.
We have

number of $(n-k)$-subsets of $[n] = \binom{n}{n-k}$,
number of $(n-k)$-subsets of $[n]$ containing a particular t subset of $[n]$ where $t \le n-k = \binom{n-t}{n-k-t}$, and
number of t weight codewords of $C = A_t$.

Therefore, number of $(n-k)$-subsets of $[n]$ which do not contain any codeword of $C \ge \binom{n}{n-k} - \sum_{t=d}^{n-k} A_t\binom{n-t}{n-k-t}$. Using (3.11), there exists a $(n-k)$-subset of $[n]$ that does not contain any codeword of C.

Let $Y \subset [n]$ with $|Y| = n-k$ be such that it does not contain support of any codeword of C. Let $\pi : [s] \to \mathbb{N}$ be a label such that $\pi(1) + \pi(2) + \cdots + \pi(s) = n$. Without loss of generality, assume $\Gamma^{(i)}(P) = \{i\}$ for $i \in P, 1 \le i \le r$ such that $V_1 \oplus V_2 \oplus \cdots \oplus V_r = V_Y$. Therefore, $\dim V_1 + \dim V_2 + \cdots + \dim V_r = n-k$ and $V_1 \oplus V_2 \oplus \cdots \oplus V_r$ does not contain any codeword of the code C.

First, we show that (P,π)-balls of radius r centered at the codewords of C cover $\mathbb{F}_q^n$. Using (2.1) for $1 \le r \le n-k$, we have

$$\begin{aligned}
|B_{(P,\pi)}(0,r)| &= 1 + \sum_{i=1}^{r} \sum_{I \in \theta_1(i)} (q^{k_i} - 1) \prod_{l<i} q^{k_l} \\
&= 1 + (q^{k_1} - 1) + (q^{k_2} - 1)q^{k_1} + \cdots + (q^{k_r} - 1)q^{k_1} q^{k_2} \cdots q^{k_{r-1}} \\
&= q^{k_1 + k_2 + \cdots + k_r} \\
&= q^{n-k}.
\end{aligned}$$

Thus, we get $|C| \cdot |B_{(P,\pi)}(0,r)| = q^n$. Now we prove that

$$B_{(P,\pi)}(0,r) \cap B_{(P,\pi)}(c,r) = \phi \quad \text{for all } c(\neq 0) \in C.$$

Else, there exists $x \in \mathbb{F}_q^n$ such that $x \in B_{(P,\pi)}(0,r) \cap B_{(P,\pi)}(c,r)$. Now $x \in B_{(P,\pi)}(0,r)$ implies that $w_{(P,\pi)}(x) \le r$, which gives us $x \in V_1 \oplus V_2 \oplus \cdots \oplus V_r$. Also,

$x \in B_{(P,\pi)}(c,r)$ implies that $w_{(P,\pi)}(c-x) \leq r$, which gives us $c-x \in V_1 \oplus V_2 \oplus \cdots \oplus V_r$. Combining the two, we get $c \in V_1 \oplus V_2 \oplus \cdots \oplus V_r$, which is a contradiction. Thus C is an r-perfect code. □

We illustrate Theorem 3.3 with the help of a few examples.

Example 3.1 (The Extended Binary Hamming Code). The weight enumerator polynomial [11] of the extended binary Hamming code $(16, 11, 4)_H$ is given by

$$1 + 140x^4 + 448x^6 + 870x^8 + 448x^{10} + 140x^{12} + x^{16}.$$

We have

$$\binom{16}{5} - 140\binom{12}{1} = 4368 - 140 \times 12 = 2688 > 0.$$

Therefore, for each r, $1 \leq r \leq n-k$, there exists a poset block structure (P, π) such that the extended binary Hamming code $(16, 11, 4)_H$ is an r-perfect (P, π)-code. The weight enumerator polynomial of the extended binary Hamming code $(32, 26, 4)_H$ is given by

$$\begin{aligned} 1 &+ 1240x^4 + 27776x^6 + 330460x^8 + 2011776x^{10} + 7063784x^{12} + 14721280x^{14} \\ &+ 18796230x^{16} + 14721280x^{18} + 7063784x^{20} + 2011776x^{22} + 330462x^{24} \\ &+ 27776x^{26} + 1240x^{28} + x^{32}. \end{aligned}$$

We have

$$\binom{32}{6} - 1240\binom{28}{2} - 27776\binom{26}{0} = 906192 - 1240 \times 378 - 27776 = 409696 > 0.$$

Therefore, for each r, $1 \leq r \leq n-k$, there exists a poset block structure (P, π) such that the extended binary Hamming code $(16, 11, 4)_H$ is an r-perfect (P, π)-code. In general, for the extended binary Hamming code $(2^m, 2^m - m - 1, 4)_H$, if

$$\binom{2^m}{m+1} - A_4\binom{2^m - 4}{m-3} - \cdots - A_{m+1} > 0$$

then for each r, $1 \leq r \leq n-k$, there exists a poset block structure (P, π) such that the extended binary Hamming code $(2^m, 2^m - m - 1, 4)_H$ is r-perfect (P, π)-code.

Example 3.2 (Double Circulant code). The weight enumerator polynomial [11] of the Karlin's double circulant code $(28, 14, 8)_H$ is given by

$$1 + 546x^8 + 1456x^{10} + 3549x^{12} + 5280x^{14} + 3549x^{16} + 1456x^{18} + 546x^{20} + x^{28}.$$

We have

$$\begin{aligned}&\binom{28}{14} - 546\binom{20}{6} - 1456\binom{18}{4} - 3549\binom{16}{2} - 5280\\ &\quad = 40116600 - 546 \times 38760 - 1456 \times 3060 - 3549 \times 120 - 5280\\ &\quad = 14067120 > 0.\end{aligned}$$

Therefore, for each r, $1 \leq r \leq n-k$, there exists a poset block structure (P, π) such that the Karlin's double circulant code $(28, 14, 8)_H$ is an r-perfect (P, π)-code.

Example 3.3 (BCH Code). The weight enumerator polynomial [11] of the BCH code $(32, 21, 6)$ is given by

$$\begin{aligned}&1 + 992x^6 + 10540x^8 + 60512x^{10} + 228160x^{12} + 446400x^{14} + 603942x^{16}\\ &\quad + 446400x^{18} + 228160x^{20} + 60512x^{22} + 10540x^{24} + 992x^{26} + x^{32}.\end{aligned}$$

We have

$$\begin{aligned}&\binom{32}{11} - 992\binom{26}{5} - 10540\binom{24}{3} - 60512\binom{22}{1}\\ &\quad = 129024480 - 992 \times 65780 - 10540 \times 2024 - 60512 \times 22\\ &\quad = 41106496 > 0.\end{aligned}$$

Therefore, for each r, $1 \leq r \leq n-k$, there exists a poset block structure (P, π) such that the BCH code $(32, 21, 6)_H$ is an r-perfect (P, π)-code.

4. Conclusion

With the advent of poset block metric, the study of perfect codes gained momentum in a new direction. Instead of searching for perfect codes with respect to a particular metric, we now look for poset block structures that turn a given code into a perfect code. As mentioned in [2], the pair (P, π) can be identified with a quoset (quasi ordered set) whose number is known to grow exponentially with respect to n, the length of the code. Therefore, more often than not we can expect a given code to be perfect with respect to some suitably chosen poset block metric. We have explicitly characterized all poset block structures which turn a given code into a 1-perfect code. A characterization of r-perfect poset block code has also been given. Given the Hamming weight enumerator of a code, the construction of a poset block structure to make a code r-perfect code has been given.

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