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Highly accurate compact difference scheme for fourth order parabolic equation with Dirichlet and Neumann boundary conditions: Application to good Boussinesq equation

Deepti Kaur[a], R.K. Mohanty[b,*]

[a] Department of Mathematics, Maitreyi College, University of Delhi, New Delhi 110021, India
[b] Department of Applied Mathematics, Faculty of Mathematics and Computer Science, South Asian University, Akbar Bhawan, Chanakyapuri, New Delhi 110021, India

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ABSTRACT

In this work, a three-level implicit compact difference scheme for the generalised form of fourth order parabolic partial differential equation is developed. The discretization is derived by approximating the lower order derivative terms using the governing differential equation with the imbedding technique and is fourth order accurate in space and second order accurate in time. The current approach is advantageous since the boundary conditions are completely satisfied and no further approximations are required to be carried out at the boundaries. The ability of the proposed scheme in handling linear singular problems is examined. The value of first order space derivative is computed alongwith the solution so it does not have to be estimated using the calculated value of the solution. The method successfully works for the highly nonlinear good Boussinesq equation for which more accurate solutions are obtained for the single and the double-soliton solutions in comparison with the existing numerical methods.

1. Introduction

In the present study, the following class of parabolic partial differential equations (PDEs) of fourth order is considered

$$A(x,t)\frac{\partial^4 y}{\partial x^4}+\frac{\partial^2 y}{\partial t^2}=k(x,t,y,y_t,y_x,y_{xx},y_{xxx}),\qquad (x,t)\in\Omega \tag{1}$$

equipped with the initial conditions

$$y(x,0)=y_0(x),\quad \alpha\le x\le\beta, \tag{2.1}$$

$$\frac{\partial y}{\partial t}(x,0)=y_1(x),\quad \alpha\le x\le\beta \tag{2.2}$$

* Corresponding author.
E-mail address: rmohanty@sau.ac.in (R.K. Mohanty).

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together with the following Dirichlet and Neumann boundary conditions

$$y(\alpha,t)=\phi_0(t),\qquad \frac{\partial y}{\partial x}(\alpha,t)=\psi_0(t),\quad t>0, \tag{3.1}$$

$$y(\beta,t)=\phi_1(t),\qquad \frac{\partial y}{\partial x}(\beta,t)=\psi_1(t),\quad t>0. \tag{3.2}$$

The solution region is defined as $\Omega\equiv\{(x,\,t)|\,\alpha<x<\beta,t>0\}$ and k, y_0, y_1, ϕ_0, ϕ_1, ψ_0, ψ_1 are functions of sufficient smoothness. The solution $y(x,\,t)$ is assumed to be sufficiently smooth to preserve the accuracy and order of the underlying difference method.

The class of PDEs represented by (1) is a generalization of several significant mathematical physics equations. This class comprises, particularly the classical Euler-Bernoulli beam equation describing the displacement of a beam [28]:

$$\rho A(x)\frac{\partial^2 y}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y}{\partial x^2}\right)=k(x,t),\quad 0<x<l,\ \ t>0, \tag{4}$$

where ρ is the density of beam, $A(x)$ represents area of the cross-section of beam and $EI(x)>0$ is called the bending stiffness or flexural rigidity of the beam. It is one of the earliest still frequently used model for the dynamics of a straight elastic beam having length l for which x-axis is along the neutral axis of the beam and is derived using the Hooke's law and other simplifying assumptions. Boundary conditions associated with Eq. (4) depends on how the ends of the beam are supported. The boundary conditions of the kind (3.1) and (3.2) correspond to a beam clamped at each end. The solution of this type of parabolic PDE of fourth order with variable coefficient has various significant applications in engineering varying from the development of supplementary framework or poles in railway production to the layout of robotic models. A thorough development of the underlying equation of motion can be found in [11,18]. The numerical solution of (4) has attracted the attention of many researchers in the past and this has resulted in various methods for solving it. Earlier, Smith et al. [27] proposed a fully sinc-Galerkin approach in time and space for the fourth order parabolic PDEs with fixed and cantilever boundary conditions. Later, Mohammadi [19] derived a numerical technique utilizing sextic B-spline to solve the fourth order time-dependent PDE (4) equipped with fixed and cantilever boundary conditions. El-Gamel [9] used sinc-Galerkin method for the variable coefficient fourth order PDE in one space variable arising in studying the transverse vibrations of a homogenous flexible beam. Recently, unconditionally stable implicit two-level difference formula with second order temporal accuracy and fourth order spatial accuracy employing three-point compact stencil for solving vibration problem in one space dimension with simply supported boundary conditions is presented in [22].

Another classical model is the good Boussinesq nonlinear equation belonging to the KdV class of equations derived by Boussinesq [1] which depicts shallow water waves propagating in both the directions and is of the form

$$\frac{\partial^2 y}{\partial t^2}=\frac{\partial^2 y}{\partial x^2}+\frac{\partial^2(y^2)}{\partial x^2}-\frac{\partial^4 y}{\partial x^4},\qquad (x,t)\in\Omega. \tag{5}$$

The nonlinear strings are modeled by the above equation. In the past few years, much attention has been devoted to Eq. (5) since it possesses solutions in the form of pulses, whose shapes and velocities are preserved after interacting among themselves. These solutions are termed as solitons. Solitons have considerable physical relevance, for instance, in the study of the water waves, in dislocation theory of crystals, plasma and fluid dynamics, laser and fiber optics etc. and hence the soliton producing Eq. (5) is of great significance to the theoretical physicists and applied mathematicians. The KdV equation admit waves propagating in single direction only whereas the good Boussinesq equation governs wave solitons traveling in both right and left directions and is shown to have an extremely complicated mechanism for the solitary wave interaction in [17]. There is rich literature on numerics for the good Boussinesq equation and is still an active research field that corresponds to a number of research papers in the subject. Intially, Bratsos [2] utilized the method of lines for transforming the initial-boundary value problem associated with the nonlinear Boussinesq equation into a first order nonlinear initial value problem. El-Zoheiry [10] analyzed the soliton interaction mechanism of Eq. (5) using an iterative finite difference scheme. Ismail and Bratsos [13] derived a numerical scheme which is of order four in time and two in space for the Boussinesq equation using a predictor-corrector pair for solving the resulting nonlinear scheme in an explicit behavior. Bratsos et al. [5] presented a parametric finite difference scheme by properly linearizing the nonlinear term for solving the one-dimensional Boussinesq equation. Later, a predictor-corrector scheme was applied to a nonlinear method by employing the rational approximants to the matrix exponential term in a three time-level recurrence relation in [3]. Bratsos [4] proposed an implicit three time-level finite difference method based on rational approximants of second order using a predictor-corrector scheme for solving numerically the improved Boussinesq equation. In the recent years, Mohebbi and Asgari [25] gave three fast and high accuracy numerical schemes for solving the Boussinesq equation using discrete Fourier transform in space and fourth-order time-stepping methods for solving the resulting system of ordinary differential equations. Dehghan and Salehi [8] combined the meshfree analog equation method with the boundary knot method for the solution of the good and bad versions of the classical Boussinesq equation in one dimension. A radial basis function approximation method is implemented for the numerical solution of the good Boussinesq equation in [29]. Cai and Wang [6] derived a set of local structure-preserving algorithms for the good Boussinesq equation, comprising multisymplectic geometric structure-preserving algorithms, local energy-preserving algorithms and local momentum-preserving algorithms. Ismail and Mosally [14] derived a fourth order in space a finite difference scheme for the good Boussinesq equation by transforming it into a first order differential system in time. A fully discrete Fourier pseudospectral scheme for the Eq. (5) with second-order temporal accuracy has been proposed and analyzed for stability and convergence in [7]. Recently, Yan and Zhang [30] proposed two energy conserving schemes for the good Boussinesq equation by employing the Hamiltonian Boundary Value and Fourier pseudospectral methods. An operator splitting scheme of order two for the Eq. (5) with Fourier pseudo-spectral approximation in space has been analyzed in [31].

The classical finite difference methods to the fourth order PDEs are second order accurate with respect to the spatial discretization. In this case, the boundary approximation used affects the accuracy of the difference method. In one dimension, the standard centered difference approximation of the biharmonic operator involves five grid points. Hence, a significant drawback of the conventional procedure for higher order problems is the extension of the computational cell as the order of the approximation is improved leading to more arithmetic operations. Since the size of the stencil plays a vital role in computation speed, hence it is necessary for the computation domain to be compact for making the computation efficient and reliable. A compact difference method uses nodes positioned just immediately close to the node around which differences are sought. In addition, if the method has order of accuracy higher than two, then it is called higher order compact difference method. These methods achieve high accuracy with much coarser discretization and are applicable straight away adjacent to the boundary excluding any additional points outside the boundary of the solution region. The increasing development and recognition of compact difference schemes revived interest in the discretization of fourth order problems. In that regard, Mohanty and Kaur [20,21] derived highly accurate difference formulas using Numerov kind and off-step discretization for the second kind quasi-linear parabolic PDEs of fourth order in one spatial dimension, i.e. when y and y_{xx} are specified at the boundary by decomposing the original problem into a system of two coupled second order equations using just three spatial nodes of a variable mesh which can effectively solve with high accuracy many important nonlinear equations such as the good Boussinesq equation. However, for the boundary conditions (3.1) and (3.2), i.e. when y and y_x are specified, the reduction to a system of two coupled second order equations is unattainable and in this case, we have to derive a compact difference approximation of higher order for the normal derivative. Recently, high order compact difference schemes have been derived for the one-dimensional unsteady biharmonic problem subject to Dirichlet and Neumann boundary conditions in [23,24]. In this work, employing the values of y and y_x at each mesh point, we derive implicit difference formulas for parabolic PDE (1) of fourth order with boundary conditions (3.1) and (3.2).

The remaining article is organised as follows. In Section 2, novel discretization strategies are formulated and the derivation of the scheme is discussed. The stability is discussed using the characteristic equation in Section 3 and we present compact difference scheme for a fourth order linear singular problem by reforming the derived scheme in a manner that the accuracy and order of the solution is preserved all over inclusive of the domain in the neighborhood of the singularity. Some numerical examples are provided in Section 4 for illustrating the practical application of the present method and it is revealed by the numerical results that the derived method has improved accuracy as compared with the earlier work. Lastly, some final remarks are accorded in Section 5.

2. Three-level compact difference scheme and derivation

In view of the given boundary conditions, we use the values of $y(x, t)$ and its first order space derivative $z(x,t) = y_x(x,t)$ for developing high order implicit difference scheme for the fourth order parabolic PDE (1).

Let the grid spacing in the spatial and temporal directions be denoted by $h > 0$ and $k > 0$, respectively, so that $x_l = \alpha + lh,\ l = 0,\dots,M+1$ and $t_j = jk,\ j = 0,1,\dots$; M being a positive integer and $(M+1)h = \beta - \alpha$. Let $\lambda = (k/h^2) > 0$ be the mesh ratio parameter. We superimpose the solution region Ω by a set of grid points (x_l, t_j). Let $Y_l^j = y(x_l, t_j)$ and $Z_l^j = z(x_l, t_j)$ be the exact values of $y(x, t)$ and $z(x, t)$, respectively and are approximated by y_l^j and z_l^j, respectively at the same node.

Firstly, we exemplify the technique employed for deriving the spatially fourth order accurate scheme for the PDE (1) by considering the following variable coefficient fourth order ordinary differential equation:

$$A(x)\frac{d^4y}{dx^4} = k(x), \quad a < x < b, \tag{6}$$

with y and $z = \frac{dy}{dx}$ specified at the boundary. Using the values of y and z at simply three mesh points of a compact cell, the Taylor series yield the following approximations:

$$h^4\left(\frac{d^4Y}{dx^4}\right)_l = -12(Y_{l-1} - 2Y_l + Y_{l+1}) + 6\ h(Z_{l+1} - Z_{l-1}) - \frac{h^6}{15}\left(\frac{d^6Y}{dx^6}\right)_l + O(h^8), \tag{7.1}$$

$$h^5\left(\frac{d^5Y}{dx^5}\right)_l = -90(Y_{l+1} - Y_{l-1}) + 30\ h(Z_{l-1} + 4Z_l + Z_{l+1}) + O(h^7), \quad l = 1,\dots,M \tag{7.2}$$

With the aim to derive the fourth order accurate finite difference scheme for the differential Eq. (6), consider the following linear combination:

$$h^4(P_0k_l + hP_1k_{x_l} + h^2P_2k_{xx_l}) = \left[P_0A_l + hP_1A_{x_l} + h^2P_2A_{xx_l}\right]h^4\left(\frac{d^4Y}{dx^4}\right)_l + \left[P_1A_l + 2hP_2A_{x_l}\right]h^5\left(\frac{d^5Y}{dx^5}\right)_l + P_2A_lh^6\left(\frac{d^6Y}{dx^6}\right)_l \tag{8}$$

where $k_l = k(x_l)$, $A_l = A(x_l)$ and so on. The parameters P_0, P_1, P_2 have to be found appropriately. Applying the approximation (7.1) in (8), we have

$$\begin{aligned} &h^4(P_0k_l + hP_1k_{x_l} + h^2P_2k_{xx_l}) \\ &\quad = \left[P_0A_l + hP_1A_{x_l} + h^2P_2A_{xx_l}\right]\left[-12(Y_{l-1} - 2Y_l + Y_{l+1}) + 6\,h(Z_{l+1} - Z_{l-1}) - \frac{h^6}{15}\left(\frac{d^6Y}{dx^6}\right)_l\right] \\ &\qquad + \left[P_1A_l + 2hP_2A_{x_l}\right]h^5\left(\frac{d^5Y}{dx^5}\right)_l + P_2A_lh^6\left(\frac{d^6Y}{dx^6}\right)_l + O(h^8) \end{aligned} \tag{9}$$

Equating the coefficients of $h^5\left(\frac{d^5Y}{dx^5}\right)_l$ and $h^6\left(\frac{d^6Y}{dx^6}\right)_l$ to zero in (9), we obtain

$$P_1 = -2h\frac{A_{x_l}}{A_l}P_2, \quad P_0 = 15P_2$$

Let $P_2 = 1$, so $P_0 = 15$ and $P_1 = -2h\frac{A_{x_l}}{A_l}$. Using the above parameter values and the below approximation in (9):

$$k_{x_l} = \frac{k_{l+1} - k_{l-1}}{2h} + O(h^2), \tag{10.1}$$

$$k_{xx_l} = \frac{k_{l-1} - 2k_l + k_{l+1}}{h^2} + O(h^2), \tag{10.2}$$

the following fourth order difference method for the Eq. (6) is obtained:

$$\begin{aligned} &\left[A_l - \frac{2h^2}{15}\frac{A_{x_l}}{A_l}A_{x_l} + \frac{h^2}{15}A_{xx_l}\right][-2(Y_{l-1} - 2Y_l + Y_{l+1}) + h(Z_{l+1} - Z_{l-1})] \\ &\quad = \frac{h^4}{90}\left[\left(1 + h\frac{A_{x_l}}{A_l}\right)k_{l-1} + 13k_l + \left(1 - h\frac{A_{x_l}}{A_l}\right)k_{l+1}\right] + O(h^8), \quad l = 1, \ldots, M \end{aligned} \tag{11}$$

In order to get the fourth order approximation for $z(x)$, we consider

$$\begin{aligned} &h^4(S_0k_l + hS_1k_{x_l} + h^2S_2k_{xx_l}) \\ &= \left[S_0A_l + hS_1A_{x_l} + h^2S_2A_{xx_l}\right]h^4\left(\frac{d^4Y}{dx^4}\right)_l + \left[S_1A_l + 2hS_2A_{x_l}\right]h^5\left(\frac{d^5Y}{dx^5}\right)_l + S_2A_lh^6\left(\frac{d^6Y}{dx^6}\right)_l \end{aligned} \tag{12}$$

where S_0, S_1 and S_2 are the unknown parameters. With the help of the approximation (7.2) in (12), we obtain

$$\begin{aligned} &h^4(S_0k_l + hS_1k_{x_l} + h^2S_2k_{xx_l}) \\ &\quad = \left[S_0A_l + hS_1A_{x_l} + h^2S_2A_{xx_l}\right]h^4\left(\frac{d^4Y}{dx^4}\right)_l \\ &\quad + \left[S_1A_l + 2hS_2A_{x_l}\right][-90(Y_{l+1} - Y_{l-1}) + 30\,h(Z_{l-1} + 4Z_l + Z_{l+1})] + S_2A_lh^6\left(\frac{d^6Y}{dx^6}\right)_l + O(h^7) \end{aligned} \tag{13}$$

Equating the coefficients of $h^4\left(\frac{d^4Y}{dx^4}\right)_l$ and $h^6\left(\frac{d^6Y}{dx^6}\right)_l$ to zero in (13), we get

$$S_0A_l + hS_1A_{x_l} + h^2S_2A_{xx_l} = 0, \quad S_2 = 0$$

Let $S_1 = 1$, we get $S_0 = -h\frac{A_{x_l}}{A_l}$. Using the approximation (10.1) and the obtained value of the parameters in (13), we obtain the following discretization for approximating z with $O(h^4)$ accuracy:

$$A_l[-3(Y_{l+1} - Y_{l-1}) + h(Z_{l-1} + 4Z_l + Z_{l+1})] = \frac{h^4}{60}\left[k_{l+1} - 2h\frac{A_{x_l}}{A_l}k_l - k_{l-1}\right] + O(h^7), \qquad l = 1, \ldots, M \tag{14}$$

The discretizations (11) and (14) can now be extended for the time-dependent case involving high order partial derivatives by defining suitable approximations to $y, z, y_t, y_{tt}, y_{xx}, z_{xx}$ at the node $(x_l,\, t_j)$ and the adjoining nodes $(x_{l\,\pm\,1},\, t_j)$.

The following approximations for deriving the high accuracy method are used. For $p = 0, \pm 1$, let

$$\overline{Y}^j_{l+p} = \tau Y^{j+1}_{l+p} + (1 - 2\tau)Y^j_{l+p} + \tau Y^{j-1}_{l+p}, \tag{15.1}$$

$$\overline{Z}^j_{l+p} = \tau Z^{j+1}_{l+p} + (1 - 2\tau)Z^j_{l+p} + \tau Z^{j-1}_{l+p}, \tag{15.2}$$

$$\overline{Y}_{t_{l+p}}^{j} = (Y_{l+p}^{j+1} - Y_{l+p}^{j-1})/(2k), \tag{15.3}$$

$$\overline{Y}_{tt_{l+p}}^{j} = (Y_{l+p}^{j+1} - 2Y_{l+p}^{j} + Y_{l+p}^{j-1})/k^2, \tag{15.4}$$

$$\overline{Y}_{xx_l}^{j} = (\overline{Z}_{l+1}^{j} - \overline{Z}_{l-1}^{j})/(2\ h), \tag{15.5}$$

$$\overline{Z}_{xx_l}^{j} = (\overline{Z}_{l+1}^{j} - 2\overline{Z}_{l}^{j} + \overline{Z}_{l-1}^{j})/h^2. \tag{15.6}$$

With the help of the above approximations, we define

$$\begin{aligned}\bar{K}_l^j &= k\Big(x_l, t_j, \bar{Y}_l^j, \bar{Y}_{t_l}^j, \bar{Z}_l^j, \bar{Y}_{xx_l}^j, \bar{Z}_{xx_l}^j\Big)\\ &= K_l^j + O\Big(k^2 + h^2\Big),\end{aligned} \tag{16}$$

where $K_l^j = k(x_l, t_j, Y_l^j, Y_{t_l}^j, Z_l^j, Y_{xx_l}^j, Z_{xx_l}^j)$. For $S = B, C, D, Y, Z,$ we denote

$$S_{pq} = \frac{\partial^{p+q} S}{\partial x^p \partial t^q}, \quad p, q = 0, 1, 2, \ldots$$

at the node $(x_l,\ t_j)$.

Further, the following approximations are considered for y_{xx} and z_{xx} at the adjacent grid points $(x_{l\,\pm\,1},\ t_j)$ as:

$$\begin{aligned}\overline{Y}_{xx_{l\pm1}}^{j} &= \frac{\pm\overline{Z}_{l\mp1}^{j} \mp 4\overline{Z}_{l}^{j} \pm 3\overline{Z}_{l\pm1}^{j}}{2h}\\ &= Y_{xx_{l\pm1}}^{j} + \tau k^2 Y_{22} - \frac{h^2}{3} Y_{40} + O(k^2h^2 + h^4),\end{aligned} \tag{17.1}$$

$$\begin{aligned}\overline{Z}_{xx_{l\pm1}}^{j} &= \frac{\mp\Big(21\overline{Y}_{l\mp1}^{j} - 48\overline{Y}_{l}^{j} + 27\overline{Y}_{l\pm1}^{j}\Big)}{2h^3} + \frac{15\overline{Z}_{l\pm1}^{j} - 9\overline{Z}_{l\mp1}^{j}}{2h^2}\\ &= Z_{xx_{l\pm1}}^{j} + \tau k^2 Y_{32} - \frac{2h^2}{5} Y_{50} + O(k^2h + h^3).\end{aligned} \tag{17.2}$$

Using these approximations, we define the approximations at adjacent nodes $(x_{l\,\pm\,1},\ t_j)$ for the function values as

$$\begin{aligned}\overline{K}_{l\pm1}^{j} &= k(x_{l\pm1}, \bar{t}_j, \overline{Y}_{l\pm1}^{j}, \overline{Y}_{t_{l\pm1}}^{j}, \overline{Z}_{l\pm1}^{j}, \overline{Y}_{xx_{l\pm1}}^{j}, \overline{Z}_{xx_{l\pm1}}^{j})\\ &= K_{l\pm1}^{j} + O(k^2 + h^2).\end{aligned} \tag{18}$$

In order to attain difference method of accuracy $O(k^2 + h^4)$, we require additional approximations for the spatial derivatives of second order of the solution variables y and z at the node $(x_p,\ t_j)$, $p = 0, \pm 1$. Let

$$\overline{\overline{Y}}_{xx_l}^{j} = \frac{1}{h^2}[a_{10}\overline{Y}_{l-1}^{j} + a_{11}\overline{Y}_{l}^{j} + a_{12}\overline{Y}_{l+1}^{j}] + \frac{1}{h}[b_{10}\overline{Z}_{l-1}^{j} + b_{11}\overline{Z}_{l}^{j} + b_{12}\overline{Z}_{l+1}^{j}] \tag{19}$$

where a_{10}, a_{11}, a_{12}, b_{10}, b_{11} and b_{12} are the coefficients to be determined. With the aid of Taylor series expansion and equating the coefficients of h^r, $r = -2, -1, 0, 1, 2$ and 3 to zero, we obtain

$$a_{10} = 2,\ \ a_{11} = -4,\ \ a_{12} = 2,\ \ b_{10} = \frac{1}{2},\ \ b_{11} = 0,\ \ b_{12} = -\frac{1}{2}.$$

Hence, we have

$$\overline{\overline{Y}}_{xx_l}^{j} = \frac{2}{h^2}[\overline{Y}_{l-1}^{j} - 2\overline{Y}_{l}^{j} + \overline{Y}_{l+1}^{j}] - \frac{1}{2h}[\overline{Z}_{l+1}^{j} - \overline{Z}_{l-1}^{j}] = Y_{xx_l}^{j} + O(k^2 + h^4) \tag{20}$$

Likewise, we obtain the following approximations:

$$\overline{\overline{Y}}_{xx_{l\pm1}}^{j} = -\frac{1}{2h^2}[5\overline{Y}_{l\mp1}^{j} - 16\overline{Y}_{l}^{j} + 11\overline{Y}_{l\pm1}^{j}] \pm \frac{1}{h}[4\overline{Z}_{l\pm1}^{j} - \overline{Z}_{l\mp1}^{j}] = Y_{xx_{l\pm1}}^{j} + O(k^2 + k^2h + h^4) \tag{21}$$

Next, an $O(k^2 + h^4)$ approximation for z_{xx} is obtained. Accordingly, suppose

$$\overline{\overline{Z}}_{xx_l}^{j} = \frac{1}{h^3}[a_{20}\overline{Y}_{l-1}^{j} + a_{21}\overline{Y}_{l}^{j} + a_{22}\overline{Y}_{l+1}^{j}] + \frac{1}{h^2}[b_{20}\overline{Z}_{l-1}^{j} + b_{21}\overline{Z}_{l}^{j} + b_{22}\overline{Z}_{l+1}^{j}] \tag{22}$$

Using Taylor series expansion around $(x_l,\ t_j)$ and equalizing the coefficients of h^r, $r = -3, -2, -1, 0, 1$ and 2 to zero, we obtain $(a_{20}, a_{21}, a_{22}, b_{20}, b_{21}, b_{22}) = (-\frac{15}{2}, 0, \frac{15}{2}, -\frac{3}{2}, -12, -\frac{3}{2})$. Thus, we have

$$\overline{\overline{Z}}_{xx_l}^{j} = \frac{15}{2h^3}[\overline{Y}_{l+1}^{j} - \overline{Y}_{l-1}^{j}] - \frac{3}{2h^2}[\overline{Z}_{l-1}^{j} + 8\overline{Z}_{l}^{j} + \overline{Z}_{l+1}^{j}] = Z_{xx_l}^{j} + O(k^2 + h^4) \tag{23}$$

Similarly,

$$\begin{aligned}\bar{\bar{Z}}^{j}_{xx_{l\pm1}} &= \mp\frac{1}{2\,h^3}\left[-51\bar{Y}^{j}_{l\mp1} - 48\bar{Y}^{j}_{l} + 99\bar{Y}^{j}_{l\pm1}\right] + \frac{1}{2\,h^2}\left[15\bar{Z}^{j}_{l\mp1} + 96\bar{Z}^{j}_{l} + 39\bar{Z}^{j}_{l\pm1}\right] \\ &= Z^{j}_{xx_{l\pm1}} + O\left(k^2 + k^2h + h^4\right) \end{aligned} \tag{24}$$

The additional approximations for the second order space derivatives are subsequently used to get the functional approximation at the grid point $(x_p,\, t_j)$, $p = 0, \pm1$:

$$\overline{\overline{K}}^{j}_{l} = k(x_l, t_j, \overline{Y}^{j}_{l}, \overline{Y}^{j}_{t_l}, \overline{Z}^{j}_{l}, \overline{\overline{Y}}^{j}_{xx_l}, \overline{\overline{Z}}^{j}_{xx_l}) = K^{j}_{l} + O(k^2 + h^4), \tag{25.1}$$

$$\overline{\overline{K}}^{j}_{l\pm1} = k(x_{l\pm1}, t_j, \overline{Y}^{j}_{l\pm1}, \overline{Y}^{j}_{t_{l\pm1}}, \overline{Z}^{j}_{l\pm1}, \overline{\overline{Y}}^{j}_{xx_{l\pm1}}, \overline{\overline{Z}}^{j}_{xx_{l\pm1}}) = K^{j}_{l\pm1} + O(k^2 + k^2h + h^4). \tag{25.2}$$

Then the difference method of accuracy $O(k^2 + h^4)$ for the differential Eq. (1) may be written as

$$\begin{aligned}&\left[A^{j}_{l} - \frac{2h^2}{15}\left(\frac{A^{j}_{x_l}}{A^{j}_{l}}\right)A^{j}_{x_l} + \frac{h^2}{15}A^{j}_{xx_l}\right]\left[-2(\overline{Y}^{j}_{l-1} - 2\overline{Y}^{j}_{l} + \overline{Y}^{j}_{l+1}) + h(\overline{Z}^{j}_{l+1} - \overline{Z}^{j}_{l-1})\right] \\ &\quad + \frac{h^4}{90}\left[\left(1 + h\frac{A^{j}_{x_l}}{A_l}\right)\overline{Y}^{j}_{tt_{l-1}} + 13\overline{Y}^{j}_{tt_l} + \left(1 - h\frac{A^{j}_{x_l}}{A_l}\right)\overline{Y}^{j}_{tt_{l+1}}\right] \\ &= \frac{h^4}{90}\left[\left(1 + h\frac{A^{j}_{x_l}}{A_l}\right)\overline{\overline{K}}^{j}_{l-1} + 13\overline{\overline{K}}^{j}_{l} + \left(1 - h\frac{A^{j}_{x_l}}{A_l}\right)\overline{\overline{K}}^{j}_{l+1}\right] + \overline{T}^{j\,(1)}_{l},\; l = 1, \ldots, M,\; j = 1, 2, \ldots \end{aligned} \tag{26}$$

and the corresponding difference method of $O(kh + h^3)$ for $z(x,\, t)$ can be written as

$$\begin{aligned}&A^{j}_{l}\left[-3(\overline{Y}^{j}_{l+1} - \overline{Y}^{j}_{l-1}) + h(\overline{Z}^{j}_{l-1} + 4\overline{Z}^{j}_{l} + \overline{Z}^{j}_{l+1})\right] + \frac{h^4}{60}\left[\overline{Y}^{j}_{tt_{l+1}} - \overline{Y}^{j}_{tt_{l-1}} - 2h\left(\frac{A^{j}_{x_l}}{A_l}\right)\overline{Y}^{j}_{tt_l}\right] \\ &= \frac{h^4}{60}\left[\overline{K}^{j}_{l+1} - \overline{K}^{j}_{l-1} - 2h\left(\frac{A^{j}_{x_l}}{A_l}\right)\overline{K}^{j}_{l}\right] + \overline{T}^{j\,(2)}_{l},\; l = 1, \ldots, M,\; j = 1, 2, \ldots, \end{aligned} \tag{27}$$

where $\overline{T}^{j\,(1)}_{l} = O(k^2h^4 + h^8)$ and $\overline{T}^{j\,(2)}_{l} = O(kh^5 + h^7)$, for arbitrary τ.

Using the Taylor series expansion, we have

$$\begin{aligned}&\left[A^{j}_{l} - \frac{2h^2}{15}\left(\frac{A^{j}_{x_l}}{A^{j}_{l}}\right)A^{j}_{x_l} + \frac{h^2}{15}A^{j}_{xx_l}\right]\left[-2(Y^{j}_{l-1} - 2\Phi^{j}_{l} + Y^{j}_{l+1}) + h(Z^{j}_{l+1} - Z^{j}_{l-1})\right] \\ &= \frac{h^4}{90}\left[\left(1 + h\frac{A^{j}_{x_l}}{A_l}\right)(K^{j}_{l-1} - Y^{j}_{tt_{l-1}}) + 13(K^{j}_{l} - Y^{j}_{tt_l}) + \left(1 - h\frac{A^{j}_{x_l}}{A_l}\right)(K^{j}_{l+1} - Y^{j}_{tt_{l+1}})\right] + O(h^8) \end{aligned} \tag{28}$$

Wit the aid of the approximations (25.1) and (25.2) and the relation (28), the local truncation error $\overline{T}^{j\,(1)}_{l}$ corresponding to the difference scheme (26) is obtained as $\overline{T}^{j\,(1)}_{l} = O(k^2h^4 + k^2h^6 + h^8)$, for all choice of τ. Likewise, we observe that $\overline{T}^{j\,(2)}_{l} = O(kh^5 + h^7)$ and hence the difference formula (27) is an $O(kh + h^3)$ approximation for the spatial derivative $y_x(x, t) = z(x, t)$.

Combining with the initial and boundary conditions, the proposed compact difference formula utilizing just three grid points has the block tridiagonal matrix form that can be solved with appropriate iterative methods (see [12,15]). It is seen that the difference method (26) and (27) is of order four in space for a fixed mesh ratio parameter value $\lambda = k/h^2$.

3. Stability analysis using characteristic equation and application to linear singular problem

For discussing the stability, consider the linear fourth order PDE of the form:

$$\frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = k(x, t), \quad (x, t) \in \Omega \tag{29}$$

Applying the proposed difference method (26) and (27) of accuracy $O(k^2 + h^4)$ to the above equation yields the following difference scheme that can be written in the matrix form:

$$Su^{j+1} = (2S + T)u^{j} - Su^{j-1} + w \tag{30}$$

where

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \quad T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \quad u = \begin{bmatrix} y \\ z \end{bmatrix}, \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Here S and T are $2M \times 2M$ block tri-diagonal matrices, u is $2M$ component solution vector and w denotes the $2M$ component column vector of known boundary values and right hand side function values associated with the block system (30). The submatrices for S and T are given by:

$$S_{11} = [1, 13, 1] - 180\lambda^2\tau[1, -2, 1],\ S_{12} = 90\lambda^2\tau h[-1, 0, 1],\ S_{21} = (1 - 180\lambda^2\tau)[-1, 0, 1],$$
$$S_{22} = 60\lambda^2\tau h[1, 4, 1],\ T_{11} = 180\lambda^2[1, -2, 1],\ T_{12} = -90\lambda^2 h[-1, 0, 1],\ T_{21} = 180\lambda^2[-1, 0, 1],$$
$$T_{22} = -60\lambda^2\tau h[1, 4, 1],$$

where $[p,\ q,\ r]$ is the $M \times M$ tri-diagonal matrix with eigenvalues $q + 2\sqrt{pr}\cos(2\theta),\ \theta = (s\pi)/(2(M+1)),\ s = 1, \ldots, M$. Here, $y = (y_1, y_2, \ldots, y_M)^T$ and $z = (z_1, z_2, \ldots, z_M)^T$ are solution vectors.

The eigenvalues of S_{11}, S_{12}, S_{21} and S_{22} are given by $15 - 4(1 - 180\lambda^2\tau)\sin^2\theta,\ 180\lambda^2\tau h\cos(2\theta)i,\ 2(1 - 180\lambda^2\tau)\cos(2\theta)i$ and $120\lambda^2\tau h(3 - 2\sin^2\theta)$, respectively, where $i = \sqrt{-1}$.

Further, the eigenvalues of T_{11}, T_{12}, T_{21} and T_{22} are given by $-720\lambda^2\sin^2\theta,\ -180\lambda^2 h\cos(2\theta)i,\ 360\lambda^2\cos(2\theta)i$ and $-120\lambda^2\tau h(3 - 2\sin^2\theta)$, respectively.

For investigating the stability of differential Eq. (29), let us examine the homogenous part of the difference scheme (30), which may be expressed as

$$u^{j+1} = (2I + S^{-1}T)u^j - Iv^j, \tag{31.1}$$

$$v^{j+1} = Iu^j + 0v^j. \tag{31.2}$$

The error vectors (in the absence of round-off errors) at the jth iterate are denoted by $\varepsilon_1^j = u^j - U^j$ and $\varepsilon_2^j = v^j - V^j$, where

$$U^j = \begin{bmatrix} Y \\ Z \end{bmatrix}^j,\ V^{j+1} = U^j = \begin{bmatrix} Y \\ Z \end{bmatrix}^j,$$

Y and Z being exact solution vectors.

The error equation may be written as:

$$E^{j+1} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}^{j+1} = HE^j,$$

with the amplification matrix H given by

$$H = \begin{bmatrix} 2I + S^{-1}T & -I \\ I & 0 \end{bmatrix}.$$

The following characteristic equation is satisfied by the eigenvalue ξ of the matrix S:

$$\det\begin{bmatrix} 15 - 4(1 - 180\lambda^2\tau)\sin^2\theta - \xi & 180\lambda^2\tau h\cos(2\theta)i \\ 2(1 - 180\lambda^2\tau)\cos(2\theta)i & 120\lambda^2\tau h(3 - 2\sin^2\theta) - \xi \end{bmatrix} = 0 \tag{32}$$

The following characteristic equation is satisfied by the eigenvalue ρ of the matrix T:

$$\det\begin{bmatrix} -720\lambda^2\sin^2\theta - \rho & -180\lambda^2 h\cos(2\theta)i \\ 360\lambda^2\cos(2\theta)i & -120\lambda^2\tau h(3 - 2\sin^2\theta) - \rho \end{bmatrix} = 0 \tag{33}$$

Let ν denote the eigenvalue of $S^{-1}T$, where ξ and ρ satisfying (32) and (33) are the eigenvalues of S and T, respectively. Suppose μ be the eigenvalue of the amplification matrix H, then it satisfy the following characteristic equation:

$$\det\begin{bmatrix} 2 + \nu - \mu & -1 \\ 1 & -\mu \end{bmatrix} = 0$$

which gives

$$\mu^2 - 2W\mu + 1 = 0 \tag{34}$$

where $W = 1 + \frac{\nu}{2}$. Hence, it can be concluded that difference method (26) and (27) is stable as long as $|W| \le 1$, that is,

$$-4 \le \nu \le 0. \tag{35}$$

Using (32) and (33), the eigenvalues of S and T are given by

$$\xi_{1,2} = \frac{1}{2}\left[(a_1 + b_2) \pm \sqrt{(a_1 - b_2)^2 + 4b_1a_2}\right] \tag{36}$$

and

$$\rho_{1,2} = \frac{1}{2}\left[(a_3 + b_4) \pm \sqrt{(a_3 - b_4)^2 + 4b_3a_4}\right] \tag{37}$$

respectively, where

$$\begin{aligned}
a_1 &= 15 - 4\left(1 - 180\lambda^2\tau\right)\sin^2\theta,\\
a_2 &= 180i\lambda^2\tau h\cos(2\theta),\\
a_3 &= -720\lambda^2\sin^2\theta,\\
a_4 &= -180i\lambda^2 h\cos(2\theta),\\
b_1 &= 2i\left(1 - 180\lambda^2\tau\right)\cos(2\theta),\\
b_2 &= 120\lambda^2\tau h\left(3 - 2\sin^2\theta\right),\\
b_3 &= 360i\lambda^2\cos(2\theta),\\
b_4 &= -120\lambda^2\tau h\left(3 - 2\sin^2\theta\right).
\end{aligned}$$

Since the matrices S^{-1} and T commute each other, hence the eigenvalues of $S^{-1}T$ are given by $\frac{\rho_{1,2}}{\xi_{1,2}}$. Thus, from (35), we obtain the relation

$$\rho_{1,2} + 4\xi_{1,2} \geq 0 \tag{38}$$

The above condition (38) is satisfied for $\tau \geq \frac{1}{4}$. Hence, the scheme (30) is stable for $\tau \geq \frac{1}{4}$.

Consider the following linear fourth order singular parabolic equation in one-space dimension:

$$\nabla^4 y + \frac{\partial^2 y}{\partial t^2} \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{\alpha}{r}\frac{\partial}{\partial r}\right)^2 y + \frac{\partial^2 y}{\partial t^2} = k(r,t),\ 0 < r < 1,\ t > 0$$

or analogously,

$$\frac{\partial^4 y}{\partial r^4} + \frac{\partial^2 y}{\partial t^2} = b(r)\frac{\partial^2 z}{\partial r^2} + c(r)\frac{\partial^2 y}{\partial r^2} + d(r)z + k(r,t),\ 0 < r < 1,\ t > 0 \tag{39}$$

for

$$b(r) = \frac{-2p}{r},\quad c(r) = \frac{\alpha(2-p)}{r^2},\quad d(r) = \frac{p(p-2)}{r^3}.$$

For $p = 1$ and 2,

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{p}{r}\frac{\partial}{\partial r}$$

is the 1D Laplacian operator in cylindrical and spherical coordinates, respectively. It is difficult to solve Eq. (39) numerically since the solution declines in the region around the singularity $r = 0$. This difficulty is overcome by reforming our method in a way that the accuracy and order of the solution is preserved all over inclusive of the domain in the neighborhood of the singularity $r = 0$.

The derived difference method (26) and (27) implemented to the singular Eq. (39) yields the following difference schemes:

$$\begin{aligned}
&-2(\bar{y}_{l-1}^j - 2\bar{y}_l^j + \bar{y}_{l+1}^j) + h(\bar{z}_{l+1}^j - \bar{z}_{l-1}^j) + \frac{h^4}{90}\left[\bar{y}_{tt_{l-1}}^j + 13\bar{y}_{tt_l}^j + \bar{y}_{tt_{l+1}}^j\right]\\
&= \frac{h^4}{90}\Big[b_{l-1}\bar{\bar{z}}_{xx_{l-1}}^j + 13b_l\bar{\bar{z}}_{xx_l}^j + b_{l+1}\bar{\bar{z}}_{xx_{l+1}}^j + c_{l-1}\bar{\bar{y}}_{xx_{l-1}}^j + 13c_l\bar{\bar{y}}_{xx_l}^j + c_{l+1}\bar{\bar{y}}_{xx_{l+1}}^j\\
&\qquad + d_{l-1}\bar{z}_{l-1}^j + 13d_l\bar{z}_l^j + d_{l+1}\bar{z}_{l+1}^j + k_{l-1}^j + 13k_l^j + k_{l+1}^j\Big]
\end{aligned} \tag{40}$$

and

$$\begin{aligned}
&-3(\bar{y}_{l+1}^j - \bar{y}_{l-1}^j) + h(\bar{z}_{l-1}^j + 4\bar{z}_l^j + \bar{z}_{l+1}^j) + \frac{h^4}{60}\left[\bar{y}_{tt_{l+1}}^j - \bar{y}_{tt_{l-1}}^j\right]\\
&= \frac{h^4}{60}\Big[b_{l+1}\bar{z}_{xx_{l+1}}^j - b_{l-1}\bar{z}_{xx_{l-1}}^j + c_{l+1}\bar{y}_{xx_{l+1}}^j - c_{l-1}\bar{y}_{xx_{l-1}}^j + d_{l+1}\bar{z}_{l+1}^j - d_{l-1}\bar{z}_{l-1}^j\\
&\qquad + k_{l+1}^j - k_{l-1}^j\Big],\quad l = 1,\ldots,M,\quad j = 1,2,\ldots,
\end{aligned} \tag{41}$$

Although the difference scheme (40) and (41) is of $O(k^2+h^4)$, it include the terms b_{l-1}, c_{l-1}, d_{l-1} and k_{l-1}^j, which have the term $1/r_{l-1}$ and for $l=1$, that leads to singularity as $r_0=0$. Hence the derived method cannot be applied directly to solve singular equations in the domain $0<r<1$, that is, in the neighborhood of singularity. The present difficulty is overcome by reforming the method (40) and (41) so that the accuracy and order of the solution is preserved throughout the domain $[0<r<1]\times[t>0]$, including the neighborhood of the singularity $r=0$.

For obtaining a valid difference scheme of $O(k^2+h^4)$, the subsequent approximations are needed:

$$b_{l\pm1}=b_{00}\pm hb_{10}+\frac{h^2}{2}b_{20}+O(h^3), \tag{42.1}$$

$$c_{l\pm1}=c_{00}\pm hc_{10}+\frac{h^2}{2}c_{20}+O(h^3), \tag{42.2}$$

$$d_{l\pm1}=d_{00}\pm hd_{10}+\frac{h^2}{2}d_{20}+O(h^3), \tag{42.3}$$

$$k_{l\pm1}^j=k_l^j\pm hk_{r_l}^j+\frac{h^2}{2}k_{rr_l}^j+O(h^3), \tag{42.4}$$

Using the approximations (42.1)-(42.4) in the difference method (40) and (41) and neglecting the terms of higher-order, the following modified difference method for the singular Eq. (39) in the compact operator form is obtained:

$$\begin{aligned}
&-2\delta_x^2\bar{y}_l^j+h(2\mu_x\delta_x)\bar{z}_l^j+\frac{h^4}{90k^2}\left[\delta_t^2\left(y_{l-1}^j+13y_l^j+y_{l+1}^j\right)\right]\\
&=\frac{h^2}{90}\left[-24b_{10}+18c_{00}-4h^2c_{20}\right]\delta_x^2\bar{y}_l^j+\frac{h^2}{180}\left[15b_{00}+27h^2b_{20}+6h^2c_{10}+2h^2d_{00}\right]\delta_x^2\bar{z}_l^j\\
&+\frac{h}{180}\left[45b_{00}-75h^2b_{20}-6h^2c_{10}\right](2\mu_x\delta_x)\bar{y}_l^j+\frac{h^3}{180}\left[24b_{10}-3c_{00}+5h^2c_{20}+2h^2d_{10}\right](2\mu_x\delta_x)\bar{z}_l^j\\
&+\frac{h^4}{90}\left[-\frac{45b_{00}}{h^2}+75b_{20}+6c_{10}+15d_{00}+h^2d_{20}\right]\bar{z}_l^j+\frac{h^4}{90}\left[15k_l^j+h^2k_{rr_l}^j\right]
\end{aligned}\tag{43}$$

and the modified difference method for $z(x,t)$ written in the compact operator form is

$$\begin{aligned}
&-3(2\mu_x\delta_x)\bar{y}_l^j+h\left(\delta_x^2+6\right)\bar{z}_l^j+\frac{h^4}{60k^2}\left[\delta_t^2\left(y_{l+1}^j-y_{l-1}^j\right)\right]\\
&=-\frac{2h}{5}b_{00}\delta_x^2\bar{y}_l^j-\frac{h^2}{20}b_{10}(2\mu_x\delta_x)\bar{y}_l^j+\frac{h^3}{60}[2c_{00}+3b_{10}]\delta_x^2\bar{z}_l^j\\
&+\frac{h^2}{60}\left[12b_{00}+h^2(c_{10}+d_{00})\right](2\mu_x\delta_x)\bar{z}_l^j+\frac{h^3}{60}\left[6b_{10}+2h^2d_{10}\right]\bar{z}_l^j+\frac{h^5}{30}k_{r_l}^j
\end{aligned}\tag{44}$$

where δ_x and δ_t are the central difference operators with respect to space and time, respectively. μ_x is the average difference operator corresponding to the spatial direction. We note that the difference scheme (43) and (44) is an implicit three-level difference scheme of accuracy $O(k^2+h^4)$ and is independent of the terms of the form $1/r_{l\,\pm\,1}$, thus can be readily solved for $l=1,\dots,M$ in the domain $[0<r<1]\times[t>0]$.

4. Numerical illustrations

We express the proposed difference method in block tridiagonal matrix system and iterative methods are used for the solution of the system of coupled equations at each grid point. The subsequent form is then solved using block Gauss-Seidel iteration method for the linear differential equation whereas we have used Newton's nonlinear block Gauss-Seidel iteration method (see Hageman and Young [12], Kelly [15] and Saad [26]) for solving the nonlinear systems. In all the examples, we have taken $tol=1e-10$ for terminating the iterations. For computing the solution, we have chosen $\tau=0.5$.

We have solved the good Boussinesq equation by considering the single and double solitons. The numerical examples reveal that the difference scheme presented in the paper give better results as compared with the results of earlier research studies. Note that the difference method (26) and (27) for solving Eq. (1) is a three-level method. Using the initial conditions, y and z are known at $t=0$. To start computing, we require the values of y and z of desired accuracy at $t=k$. We use an explicit $O(k^2)$ scheme for obtaining y and z at the first time level, i.e. at $t=k$. As y and y_t are specified explicitly at $t=0$, hence all their successive tangential partial derivatives are known initially, i.e. the values of

$$\frac{\partial^s y_l^0}{\partial x^s},\ \frac{\partial^s z_l^0}{\partial x^s},\ \frac{\partial^{s+1}y_l^0}{\partial x^s\partial t},\ \frac{\partial^{s+1}z_l^0}{\partial x^s\partial t},\quad s=0,1,\dots$$

are known at $t=0$. The considered differential Eq. (1) may be re-written as:

$$\frac{\partial^2 y}{\partial t^2}=-A(x,t)\frac{\partial^4 y}{\partial x^4}+k(x,t,y,y_t,z,y_{xx},z_{xx}) \tag{45}$$

Table 1
Single-soliton results for the good Boussinesq Eq. (5), Example 1 for $A = 0.369$, $x^0 = 0$ at different times with $h = 0.1$ and $k = 0.01$.

t	Proposed Method MAE	Method [8] MAE	Method [14] MAE	Proposed Method RMS	Method [8] RMS	Method [14] RMS
10	6.2747(-10)	2.1346(-02)	1.0000(-06)	5.7224(-11)	4.0150(-06)	8.0000(-06)
20	2.3093(-09)	–	2.0000(-06)	2.2376(-10)	–	1.3000(-05)
30	3.9111(-09)	2.8142(-02)	2.0000(-06)	3.8050(-10)	4.0260(-06)	1.7000(-05)
40	6.1234(-09)	–	3.0000(-06)	6.0002(-10)	–	2.0000(-05)
50	7.0371(-09)	–	3.0000(-06)	6.8332(-10)	–	2.3000(-05)
60	8.9291(-09)	5.6516(-03)	–	8.9941(-10)	4.0011(-06)	–

Differentiating (45) with respect to x, we get

$$\frac{\partial^2 z}{\partial t^2} = -A\frac{\partial^4 z}{\partial x^4} - A_x\frac{\partial^4 y}{\partial x^4} + \frac{\partial k}{\partial x} + \frac{\partial k}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial k}{\partial y_t}\frac{\partial^2 y}{\partial x \partial t} + \frac{\partial k}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial k}{\partial y_{xx}}\frac{\partial^3 y}{\partial x^3} + \frac{\partial k}{\partial z_{xx}}\frac{\partial^3 z}{\partial x^3} \tag{46}$$

The following explicit $O(k^2)$ difference methods for obtaining y and z at $t = k$ are considered:

$$Y_l^1 = Y_l^0 + kY_{t_l}^0 + \frac{k^2}{2}Y_{tt_l}^0 + O(k^3) \tag{47}$$

$$Z_l^1 = Z_l^0 + kZ_{t_l}^0 + \frac{k^2}{2}Z_{tt_l}^0 + O(k^3) \tag{48}$$

Therefore, from the initial values and using their successive tangential derivatives, the values of $Y_{tt_l}^0$ and $Z_{tt_l}^0$ are determined from (45) and (46), respectively. Then lastly, these values are used in (47) and (48) to obtain the values of y and z at $t = k$.

Example 1 [3,6,8,14,29]**.** (*Single-soliton of Good Boussinesq Equation*)

Manoranjan et al. [16] obtained the following solitary wave solution of the good Boussinesq equation (5):

$$y(x,t) = -A\,\mathrm{sech}^2\left[\sqrt{\frac{A}{6}}(x - ct + x^0)\right] - \left(b + \frac{1}{2}\right) \tag{49}$$

where A is amplitude of the pulse positioned initially at $x = x^0$, $c = \pm\left[-2\left(b + \frac{A}{3}\right)\right]^{1/2}$ is the velocity of the solitary wave and b is an arbitrary parameter. The good Boussinesq equation is solved numerically with initial time $t_0 = 0$ and boundaries $a = -100$, $b = 140$. The analytical solution (49) corresponds to a solitary wave traveling to left or right according to the sign of the velocity c. If c is negative (positive), the solitary wave moves to the left (to the right). For numerical computation, we choose the similar physical constants as considered in [8,14]: $A = 0.369$, $b = -0.5$ and $x^0 = 0$. In Table 1, we compare the maximum absolute error (MAE) and root mean sqaure error (RMS) found by proposed method using $h = 0.1$ and $k = 0.01$ with [8,14]. Eq. (5) is solved at various time-levels with different time and space mesh sizes using the proposed method and the results are reported and compared with the available methods in Table 2. Table 2 shows that the results are accurate even for long time intervals. The 2D graphical comparison of analytical and numerical solution is displayed in Fig. 1(a) and Fig. 1(b) at $t = 60$ and $t = 90$, respectively.

Example 2 [8,25]**.** (*Double-soliton of Good Boussinesq Equation*)

In this example, we examine the double-soliton case by considering two waves. The solution corresponding to double-soliton wave can be written as [16]:

$$y(x,t) = -\sum_{i=1}^{2}\left\{A_i\,\mathrm{sech}^2\left[\sqrt{\frac{A_i}{6}}(x - c_i t + x_i^0)\right] + b_i + \frac{1}{2}\right\} \tag{50}$$

where A_i, $i = 1,2$ are the amplitudes of the pulse positioned initially at x_i^0, $i = 1,2$, $c_i = \pm\left[-2\left(b_i + \frac{A_i}{3}\right)\right]^{1/2}$; $i = 1,2$ are the velocities of the ith soliton and b_i, $i = 1,2$ are arbitrary parameters. The analytical solution (50) represents two solitary waves moving to the left or right according to the sign of the velocity c_i, $i = 1,2$. We consider the following two cases:

(i) two waves having equal amplitudes $A_1 = A_2 = 0.369$ placed initially at $x_1^0 = 0$ and $x_2^0 = 50$ and moving towards each other with velocities $-c_1 = c_2 = 0.868$ for $b_1 = b_2 = -0.5$ from time $t = 0$ to $t = 10$ for $-80 \le x \le 120$ with $h = 0.5$, $k = 0.01$. The results are reported in Table 3 and the plot of numerical solution versus analytical solution at $t = 10$ is depicted in Fig. 2.

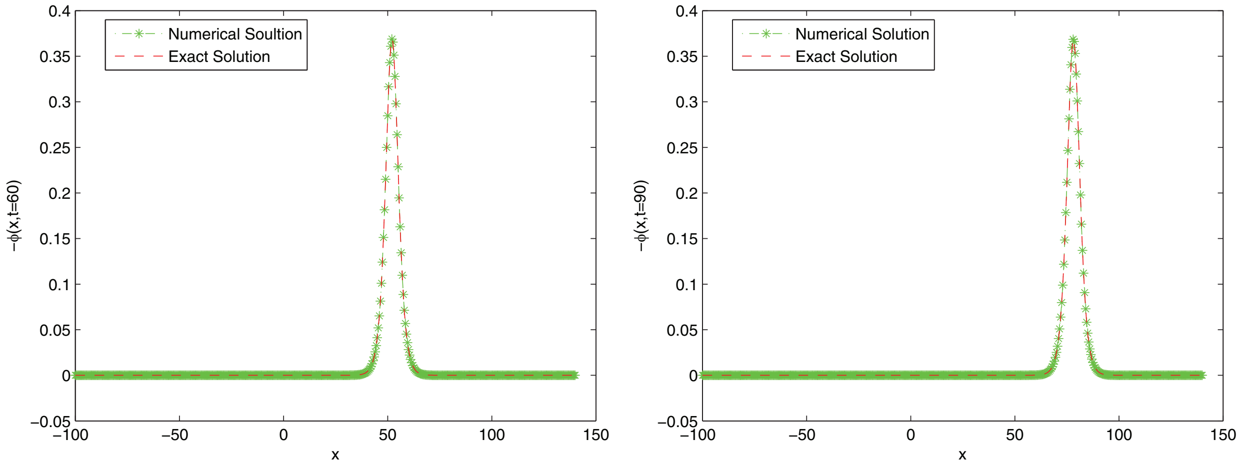
(ii) two waves of different amplitudes $A_1 = 0.2$, $A_2 = 0.4$ moving towards each other with velocities $c_1 = -0.9309$, $c_2 = 0.8563$ placed at the initial positions $x_1^0 = 0$ and $x_2^0 = 50$ for $b_1 = b_2 = -0.5$ from time $t = 0$ to $t = 10$ and the solution region is chosen as $[-80, 120]$ with different time and space steps. The results are compared with known studied method [8] in Table 4 and the plot of numerical solution versus analytical solution at $t = 10$ is depicted in Fig. 3. Also,

Table 2
Numerical results of proposed method and known studied methods for the single-soliton wave, Example 1 for $A = 0.369$, $x^0 = 0$ at different times.

Method	h	k	t	MAE
Proposed method	0.5	0.01	30	3.0611(-05)
	0.5	0.01	36	4.2160(-05)
	0.5	0.01	60	9.9559(-05)
	0.5	0.01	72	1.3824(-04)
	0.25	0.01	30	5.2234(-06)
	0.25	0.01	36	6.1865(-06)
	0.25	0.01	60	1.0936(-05)
	0.25	0.01	72	1.3923(-05)
	0.125	0.01	30	1.1931(-08)
	0.125	0.01	36	1.5767(-08)
	0.125	0.01	60	2.4517(-08)
	0.125	0.01	72	8.6236(-08)
	0.1	0.01	30	3.9111(-09)
	0.1	0.01	36	5.0313(-09)
	0.1	0.01	60	8.9291(-09)
	0.1	0.01	72	1.1444(-08)
	0.05	0.01	30	3.4708(-10)
	0.05	0.01	36	8.9887(-10)
Meshless Method [8]	–	–	30	2.8141(-02)
			36	4.7325(-03)
			60	5.6516(-03)
			72	4.8863(-04)
Linearized scheme [14]	0.1	0.001	72	9.7500(-04)
Nonlinear Method [14]	0.1	0.001	72	9.7000(-08)
Energy preserving scheme [6]	0.05	0.001	36	5.9926(-05)
	0.05	0.001	72	9.0146(-05)
Second order P-C scheme [3]	0.1	0.001	72	3.7799(-04)
	0.05	0.001	72	1.0687(-04)
Multiquadric Method [29]	0.5	0.0002	72	2.205(-05)

Table 3
Double-soliton results for the good Boussinesq Eq. (5), Example 2 (i) for $A_1 = 0.369$, $A_2 = 0.369$, $x_1^0 = 0$, $x_2^0 = 50$ at $t = 10$.

Method	h	k	MAE
Proposed method	0.5	0.1	5.5882(-03)
	0.5	0.01	5.7862(-03)
	0.25	0.1	5.6197(-03)
	0.25	0.1	5.7957(-03)
Method [8]	–	–	2.1016(-02)



(a) Plot of exact-numerical solution at $t = 60$ (b) Plot of exact-numerical solution at $t = 90$

Fig. 1. Single-soliton, plots of exact-numerical solutions at different times with $h = 0.5$ and $k = 0.01$, Example 1.

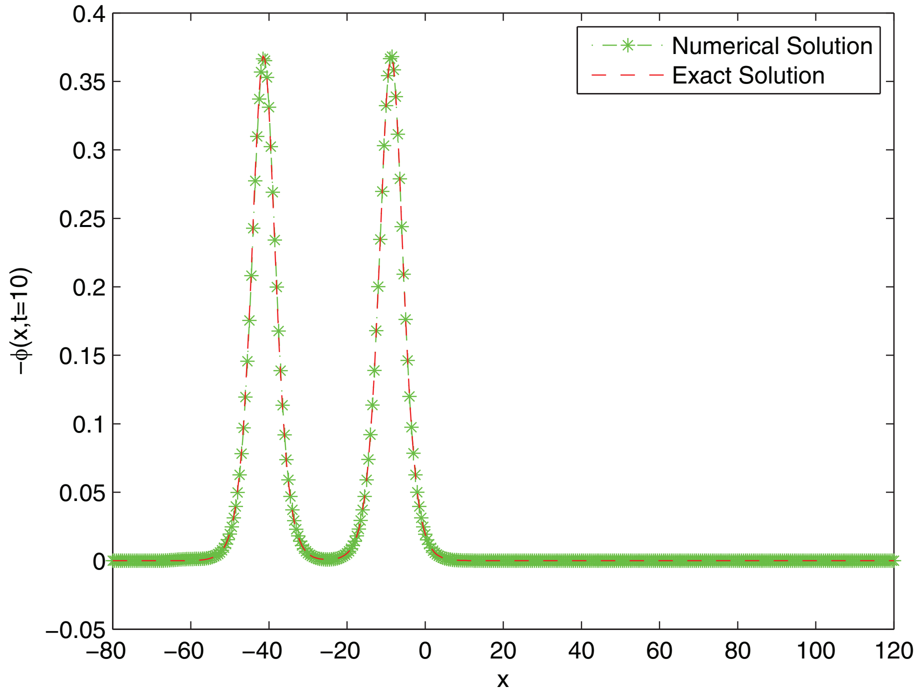


Fig. 2. Double-soliton, plot of exact-numerical solution at $t = 10$ with $h = 0.5$ and $k = 0.01$, Example 2 (i) with equal amplitudes $A_1 = A_2 = 0.369$.

Table 4
Double-soliton results for the good Boussinesq Eq. (5), Example 2 (ii) for $A_1 = 0.2$, $A_2 = 0.4$, $x_1^0 = 0$, $x_2^0 = 50$ at $t = 10$.

Method	h	k	MAE
Proposed method	0.5	0.1	2.8279(-04)
	0.5	0.01	4.9227(-05)
	0.25	0.1	2.9744(-04)
	0.25	0.01	4.9025(-05)
	0.1	0.01	7.0237(-08)
Method [8]	–	–	1.4563(-02)

Table 5
Double-soliton results for the good Boussinesq Eq. (5), Example 3 (ii) for $A_1 = 0.2$, $A_2 = 0.4$, $x_1^0 = -20$, $x_2^0 = 20$ at different times with $h = 0.1$, $k = 0.01$.

t	Proposed Method MAE		Method [25] MAE
	$h = 0.1$, $k = 0.01$	$h = 0.05$, $k = 0.01$	
0.5	1.2648(-10)	3.0643(-12)	5.7911(-09)
2.0	1.1152(-10)	3.3792(-12)	1.2203(-07)
5.0	1.7437(-09)	7.5619(-12)	1.5596(-06)
10	1.5283(-05)	5.4486(-11)	–
15	1.1107(-03)	–	1.1558(-03)

in order to compare our results with [25], we investigate the double-soliton solutions with $A_1 = 0.2$, $A_2 = 0.4$ placed at the initial positions $x_1^0 = -20$ and $x_2^0 = 20$ for $b_1 = b_2 = -0.5$. Computations are performed for $-80 \le x \le 100$ and $0 \le t \le 15$ with $h = 0.1$, $k = 0.01$ and results are presented in Table 5.

Example 3. Consider the good Boussinesq equation of the form:

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 (y^2)}{\partial x^2} - \frac{\partial^4 y}{\partial x^4} + k(x,t), \quad 0 < x < 1, \quad t > 0. \tag{51}$$

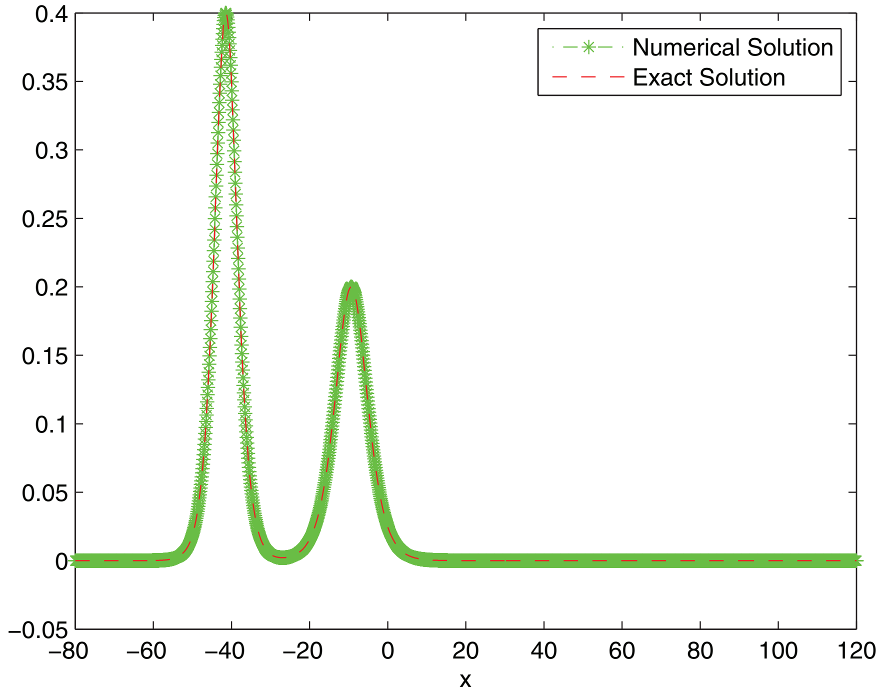


Fig. 3. Double-soliton, plot of exact-numerical solution at $t = 10$ with $h = 0.5$ and $k = 0.01$, Example 2 (ii) with different amplitudes $A_1 = 0.2$, $A_2 = 0.4$.

Table 6
The root mean square error, the maximum absolute error and the order of convergence of method (26) and (27) for Example 3 at $t = 1$ and 2 for a fixed $\lambda = (k/h^2) = 1.6$.

	$t = 1$				$t = 2$			
h	RMS	Order	MAE	Order	RMS	Order	MAE	Order
1/8	8.4272(-06)	-	1.2275(-05)	-	9.9083(-06)	-	1.4434(-05)	
1/16	5.1044(-07)	4.0452	7.6976(-07)	3.9952	5.9201(-07)	4.0649	8.9264(-07)	4.0152
1/32	3.1391(-08)	4.0233	4.8313(-08)	3.9939	3.6372(-08)	4.0247	5.5969(-08)	3.9954

Table 7
MAEs and order of convergence using the proposed $O(k^2 + h^4)$ method for Example 4 at $t = 0.5$ and $t = 1.0$.

h		$t = 0.5$	Order	$t = 1.0$	Order
1/8	ϕ	2.1273(-03)	–	5.1423(-03)	–
	ϕ_x	3.3769(-02)	–	8.2095(-03)	–
1/16	ϕ	1.6759(-04)	3.6660	4.1340(-04)	3.6368
	ϕ_x	1.5478(-03)	4.4482	3.8718(-03)	4.4062
1/32	ϕ	1.1742(-05)	3.8352	2.8306(-05)	3.8684
	ϕ_x	9.8948(-05)	3.9674	2.3878(-04)	4.0193

The analytical solution of this equation is

$$y(x,t) = e^x \sin t$$

and $k(x,t) = -[1 + 4e^x \sin t]e^x \sin t$. The numerical solution of differential Eq. (51) is evaluated using finite difference scheme (26) and (27) for $h = \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$ at time $t = 1$ and 2. We select the time length k associated with the spatial grid size h as $k = \lambda h^2$ for $\lambda = 1.6$. By means of this time length, the theoretic order of convergence turn out to be $O(h^4)$ which implies that the proposed method has fourth order spatial accuracy. In Table 6, we present the RMSs, the MAEs and the order of convergence.

Example 4 [19,27]**.** Let us consider the Euler-Bernoulli beam equation:

$$\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left((1 + \sin \pi x)\frac{\partial^2 y}{\partial x^2}\right) = k(x,t), \quad 0 < x < 1, \quad t > 0. \tag{52}$$

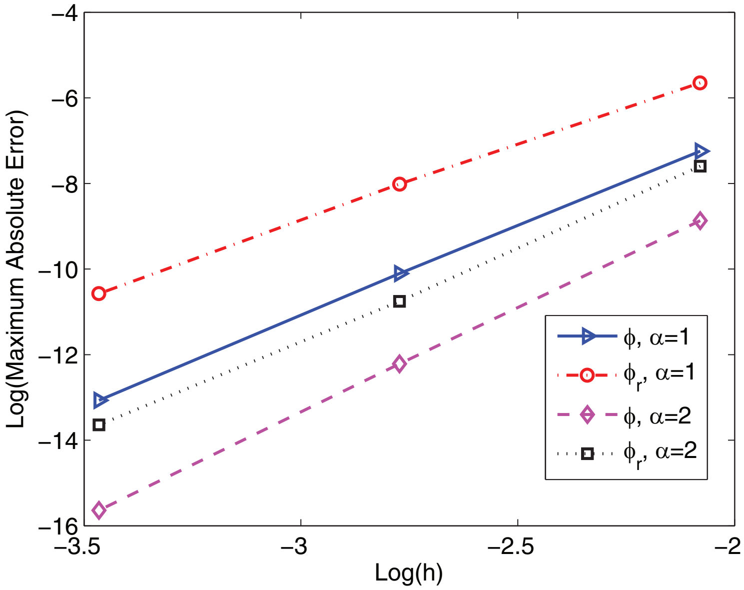


Fig. 4. Error plot at $t = 1$ for $\alpha = 1, 2$ with $\lambda = 1.6$, Example 5.

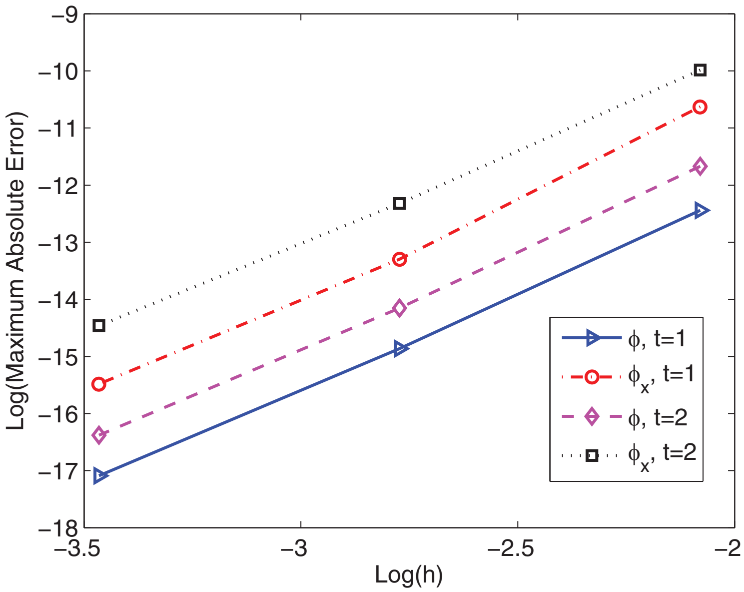


Fig. 5. Error plot at $t = 1$ and $t = 2$ for $\lambda = 1.6$, Example 6.

The analytical solution of the problem is $y(x,t) = x(1-x)e^{-t}t^2 \sin 4\pi x$ and $k(x, t)$ is consistent with the analytical solution. Using the proposed scheme (26) and (27), we report the MAEs and order of convergence for $y(x, t)$ and $y_x(x, t)$ in Table 7 at $t = 0.5$ and $t = 1$ taking $\lambda = 1.6$ with $M + 1 = 8, 16$ and 32.

Example 5. Consider the linear singular Eq. (39) having exact solution $y(r,t) = e^{-t}r^4 \sin r$. The difference method (43) and (44) is applied to compute the solution and the MAEs in y and y_r are displayed in a Log/Log scale at $t = 1$ for $\lambda = 1.6$ in Fig. 4. From Fig. 4, it is observed that the graph has nearly constant slope about four and hence the accuracy and order of the scheme (40) and (41) is preserved by using the approximations (42.1)-(42.4).

Example 6 [27]**.** We find the numerical solution of the following fourth order evolution equation:

$$\frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = k(x,t), \quad 0 < x < 1, \ \ t > 0 \tag{53}$$

equipped with the initial and boundary conditions of the form (2.1),(2.2) and (3.1),(3.2). The analytical solution is given by

$$y(x,t) = [x(1-x)]^{7/2} t^{5/2} e^{-t}.$$

and

$$k(x,t) = \frac{e^{-t}\sqrt{t}\Big(60(x-1)^4x^4 - 80t(x-1)^4x^4 + t^2\Big(105 - 3360x + 16800x^2 - 26880x^3 + 13456x^4 - 64x^5 + 96x^6 - 64x^7 + 16x^8\Big)\Big)}{16\sqrt{x(1-x)}}.$$

In this example, $k(x, t)$ has algebraic singularities at $x = 0$ and $x = 1$. The proposed difference scheme (26) and (27) is modified in a manner that the accuracy and order of the solution is maintained allover the solution space comprising the region in the neighborhood of the singularities and the MAEs in y and y_x are plotted in Fig. 5 at $t = 1, 2$ for $\lambda = 1.6$. From Fig. 5, it is evident that the graph has almost constant slope around four.

5. Conclusions

In this paper, we develop implicit difference scheme for solving the first kind 1D parabolic PDE of fourth order. The proposed difference method uses just three spatial nodes $x_l, x_{l\pm 1}$ of a single computational stencil and incorporate the given boundary conditions naturally without introducing fictitious nodes or particular schemes at the boundary. We have discussed the ability of the proposed scheme in handling linear singular problems by reforming the scheme in a manner that the accuracy and order of the method is preserved throughout the solution space inclusive of the region in the neighborhood of the singularity. The results of Examples 4 and 5 evidently illustrate that the derived method is accurate even when singularities are present at the boundaries. The numerical results illustrate the ability of the proposed scheme for solving the complex fourth order good Boussinesq equation with high accuracy. Both single and the double-soliton waves are examined and the obtained numerical results were found to have improved accuracy than the earlier known methods. For a fixed mesh ratio parameter value, fourth order convergent results are obtained as exhibited by the computed results. The solutions are displayed graphically which exhibit the agreement between the numerical and analytical solutions. Also, the first order space derivative which is of significance in various applications is computed. The technique considered here will prove useful for the formulation of stable difference schemes for fourth order evolution equations.

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References

[1] M.J. Boussinesq, Thorie de lintumescence appele onde solitaire ou de translation se propageant dans un canal rectangulaire, C.R. Acad. Sci. Paris 72 (1871) 755–759.
[2] A.G. Bratsos, The solution of the Boussinesq equation using the method of lines, Comput. Methods Appl. Mech. Eng. 157 (1998) 33–44.
[3] A.G. Bratsos, A second order numerical scheme for the solution of the one-dimensional boussinesq equation, Numer. Algor. 46 (2007) 45–58.
[4] A.G. Bratsos, A predictor-corrector scheme for the improved Boussinesq equation, Chaos Solitons Fractals 40 (2009) 2083–2094.
[5] A.G. Bratsos, C. Tsitouras, D.G. Natsis, Linearized numerical schemes for the Boussinesq equation, Appl. Numer. Anal. Comput. Math. 2 (2005) 34–53.
[6] J. Cai, Y. Wang, Local structure-preserving algorithms for the "good" Boussinesq equation, J. Comput. Phys. 239 (2013) 72–89.
[7] K. Cheng, W. Feng, S. Gottlieb, C. Wang, A fourier pseudospectral method for the "good" Boussinesq equation with second-order temporal accuracy, Numer. Methods Partial Differ. Equ. 31 (2015) 202–224.
[8] D. Mehdi, S. Rezvan, A meshless based numerical technique for traveling solitary wave solution of Boussinesq equation, Appl. Math. Model. 36 (2012) 1939–1956.
[9] M. El-Gamel, A note on solving the fourth-order parabolic equation by the Sinc-Galerkin method, Calcolo 52 (2015) 327–342.
[10] H. El-Zoheiry, Numerical investigation for the solitary waves interaction of the "good" Boussinesq equation, Appl. Numer. Math. 45 (2003) 161–173.
[11] D.G. Gorman, Free Vibrations Analysis of Beams and Shafts, Wiley, New York, 1975.
[12] L.A. Hageman, D.M. Young, Applied Iterative Methods, Dover Publication, New York, 2004.
[13] M.S. Ismail, A.G. Bratsos, A predictor–corrector scheme for the numerical solution of the Boussinesq equation, J. Appl. Math. Comput. 13 (2003) 11–27.
[14] M.S. Ismail, F. Mosally, A fourth order finite difference method for the good Boussinesq equation, Abstr. Appl. Anal. Art. ID 323260 (2014) 10.
[15] C.Y. Kelly, Iterative Methods for Linear and Non-linear Equations, SIAM Publications, Philadelphia, 1995.
[16] V.S. Manoranjan, A.R. Mitchell, J.L. Morris, Numerical solutions of the good Boussinesq equation, SIAM J. Sci. Stat. Comput. 5 (1984) 946–957.
[17] V.S. Manoranjan, T. Ortega, J.M. Sang-Serna, Soliton and antisoliton interactions in the "good" Boussinesq equation, J. Math. Phys. 29 (1988) 1964–1968.
[18] L. Meirovitch, Principles and Techniques of Vibrations, Prentice Hall, New Jersey, 1997.
[19] R. Mohammadi, Sextic b-spline collocation method for solving euler-bernoulli beam models, Appl. Math. Comput. 241 (2014) 151–166.
[20] R.K. Mohanty, D. Kaur, High accuracy implicit variable mesh methods for numerical study of special types of fourth order non-linear parabolic equations, Appl. Math. Comput. 273 (2016) 678–696.
[21] R.K. Mohanty, D. Kaur, A class of quasi-variable mesh methods based on off-step discretization for the numerical solution of fourth-order quasi-linear parabolic partial differential equations, Adv. Differ. Equ. (2016). 2016:326
[22] R.K. Mohanty, D. Kaur, Unconditionally stable high accuracy compact difference schemes for multi-space dimensional vibration problems with simply supported boundary conditions, Appl. Math. Model. 55 (2018) 281–298.
[23] D. Kaur, R.K. Mohanty, Two-level implicit high order method based on half-step discretization for 1d unsteady biharmonic problems of first kind, Appl. Numer. Math. 139 (2019) 1–14.
[24] R.K. Mohanty, D. Kaur, High accuracy two-level implicit compact difference scheme for 1d unsteady biharmonic problem of first kind: application to generalized kuramoto-sivashinsky equation, J. Differ. Equ. Appl. 25 (2019) 243–261.

[25] A. Mohebbi, Z. Asgari, Efficient numerical algorithms for the solution of "good" boussinesq equation in water wave propagation, Comput. Phys. Comm. 182 (2011) 2464–2470.

[26] Y. Saad, Iterative Methods for Sparse Linear Systems, SIAM Publisher, 2003.

[27] R.C. Smith, K.L. Bowers, J. Lund, A fully Sinc-Galerkin method for euler-bernoulli beam models, Numer. Methods Partial Differ. Equ. 8 (1992) 171–202.

[28] C.W. Soh, Euler-bernoulli beams from a symmetry standpoint-characterization of equivalent equations, J. Math. Anal. Appl. 345 (2008) 387–395.

[29] M. Uddin, S. Haq, M. Ishaq, RBF-pseudospectral method for the numerical solution of good Boussinesq equation, Appl. Math. Sci. (Ruse) 6 (2012) 2403–2410.

[30] J. Yan, Z. Zhang, New energy-preserving schemes using hamiltonian boundary value and fourier pseudospectral methods for the numerical solution of the "good" boussinesq equation, Comput. Phys. Commun. 201 (2016) 33–42.

[31] C. Zhang, H. Wang, J. Huang, C. Wang, X. Yue, A second order operator splitting numerical scheme for the "good" Boussinesq equation, Appl. Numer. Math. 119 (2017) 179–193.