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On some properties of A-nuclei and σ-A-nuclei of a quasigroup

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In this paper, we investigate the properties of A-nuclei and σ-A-nuclei of a quasigroup including connections between components of σ-A-nuclei and local identity elements. To make the study easier, we give the explicit structure of σ-A-nuclei of a loop, a unipotent left loop and a unipotent right loop in terms of translation maps. We also find the left, right and middle A-nuclei of a quasigroup which is an isostrophic image of a loop in terms of translation maps.

Keywords: Quasigroup; loop; translations; autostropy; autostrophy; nucleus; A-nuclei; σ-A-nuclei.

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1. Introduction

The concept of quasigroup nuclei was introduced by Garrison [5] in 1940. A left/ right/ middle nucleus of a quasigroup measures how far a quasigroup is from a group by measuring near-associativity of the quasigroup. Shcherbacov [11, 12] has discussed a detailed historical background of A-nuclei of a quasigroup. Pflugfelder

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[8, 11] showed that if a quasigroup $(Q, \cdot)$ has a nontrivial left/ right/ middle nucleus then the quasigroup $(Q, \cdot)$ has a left identity/ a right identity/ an identity element, respectively. Belousov [1] generalized the concept of Garrison's quasigroup nuclei on proper quasigroup by introducing the notion of left, right and middle regular permutations which can be used to measure the near-associativity of quasigroups that neither have left nor right unit element. The connections between quasigroup nuclei and the groups of regular permutations of quasigroups have been studied in [1–3, 7, 12]. Later left, right and middle A-nuclei (autotopy nuclei) of a quasigroup is defined as groups of left, right and middle regular permutations, respectively, and studied by many mathematicians [1, 6, 7, 11, 12]. The concept of A-nuclei can be considered as generalization of nuclei of a quasigroup in the sense that when the quasigroup is a loop, the (nontrivial) components of left/right A-nucleus of the loop coincide with left/ right translations by elements of the left/right nuclei, respectively. For a more detailed survey on A-nuclei of a quasigroup the reader may refer [11, 12].

The left, right and middle σ-A-nuclei of a quasigroup $(Q, \cdot)$ have been defined in [12] as an autostrophy of the quasigroup of the form $(\sigma, (\alpha, \beta, \gamma))$ for the values $\beta = \varepsilon$, $\alpha = \varepsilon$ and $\gamma = \varepsilon$, respectively, where α, β, γ are permutations of the set Q, ε is the identity permutation and $\sigma \in S_3$. It may be noted that this definition is not a generalization of that of A-nuclei. So in this paper, we redefine left, right and middle σ-A-nuclei of a quasigroup as a generalization of left, right and middle A-nuclei and study their properties. Also to make the study easier, we explicitly derive structure of σ-A-nuclei of a loop, a unipotent left loop and a unipotent right loop in terms of translation maps. This will reduce the time required in computation of σ-A-nuclei of the mentioned quasigroups. Further, we find A-nuclei of a quasigroup which is an isostrophic image of a loop in terms of translation maps.

The paper has the following structure: Section 2 presents some basic definitions, notations and results required for Sec. 3. In Sec. 3, we give definition of σ-A-nuclei of a quasigroup and connections between components of σ-A-nuclei and local identity elements of a quasigroup. Further, we give an explicit structure of A-nuclei of an isostrophic image of a loop in terms of translation maps. Section 4 concludes this paper.

2. Preliminaries

In this section, we give some definitions, notations and results which are required for the results in the subsequent section [3, 4, 8–12].

Definition 2.1. Let $(Q, *)$ be a binary groupoid. $(Q, *)$ is called a *quasigroup* if there exist unique solutions $x, y \in Q$ of the equations $x * a = b$ and $a * y = b$ for all ordered pair $(a, b) \in Q^2$.

For any given quasigroup $(Q, *)$, the operations $/, \backslash, \cdot, //$ and $\backslash\backslash$ on the set Q are as follows:

$x/y = z \Leftrightarrow z * y = x$ (right division),

$x\backslash y = z \Leftrightarrow x * z = y$ (left division),
$x \cdot y = z \Leftrightarrow y * x = z$ (opposite multiplication),
$x//y = z \Leftrightarrow y/x = z \Leftrightarrow z * x = y$ (opposite right division)
and $x\backslash\backslash y = z \Leftrightarrow y\backslash x = z \Leftrightarrow y * z = x$ (opposite left division).

The set Par($*$)=$\{*, /, \backslash, \cdot, //, \backslash\backslash\}$ is known as the set of *parastrophes* of quasigroup operation $*$.

Note that the parastrophes $/, \backslash, \cdot, //$ and $\backslash\backslash$ are sometimes denoted as $*^{(13)}$, $*^{(23)}$, $*^{(12)}$, $*^{(123)}$ and $*^{(132)}$, respectively.

Let $(G, \cdot)$ be a groupoid and let a be a fixed element in G. The *left and the right translations* denoted by L_a and R_a are, respectively, defined as $L_a x = a \cdot x$ and $R_a x = x \cdot a$ for all $x \in G$.

If $(G, \cdot)$ is a quasigroup then there exist one more translation known as *middle translation* denoted by P_a and is defined as $x \cdot P_a x = a$ for all $x \in G$.

Definition 2.2. An element $f(x)$ $(e(x), s(x))$ of a quasigroup $(Q, \cdot)$ is called a *left* (*right, middle*) *local identity element* of an element $x \in Q$, if $f(x) \cdot x = x$ $(x \cdot e(x) = x, x \cdot x = s(x))$.

An element $f \in Q$ $(e \in Q, s \in Q)$ is called a *left* (*right, middle*) *identity element* of a quasigroup $(Q, \cdot)$ if $f(x) = f\ \forall x \in Q$ $(e(x) = e \forall x \in Q, s(x) = s\ \forall x \in Q)$. A quasigroup $(Q, \cdot)$ with a left, right and middle identity elements $f \in Q$, $e \in Q$ and $s \in Q$ are called a *left loop, a right loop and a unipotent loop,* respectively. A quasigroup $(Q, \cdot)$ with an identity element $e \in Q$ (left and right identity elements coincide) is called a *loop*.

A quasigroup need not have a left/right/middle identity element, but from the definition of quasigroup we can definitely say that every element of a quasigroup has (a unique) left, right and middle local identity elements.

Table 1 shows connections between local identity elements in all the 6 parastrophes of a quasigroup [12].

Definition 2.3. Let $(G, \circ)$ and $(G, \cdot)$ be two binary groupoids. Then $(G, \circ)$ is an *isotopic image* of $(G, \cdot)$ (or $(G, \circ)$ is an *isotope* of $(G, \cdot)$), if there exist permutations $\alpha_1, \alpha_2, \alpha_3$ of the set G such that

$$x \circ y = \alpha_3^{-1}(\alpha_1 x \cdot \alpha_2 y) \tag{2.1}$$

for all $x, y \in G$.

Table 1. Connections between local identity elements of parastrophic quasigroups.

	ε	(12)	(13)	(23)	(132)	(123)
f	f	e	s	f	e	s
e	e	f	e	s	s	f
s	s	s	f	e	f	e

The above equality can be written as $(G, \circ) = (G, \cdot)T$, where the ordered 3 tuple $T = (\alpha_1, \alpha_2, \alpha_3)$ is known as isotopy of binary groupoid $(G, \cdot)$.

If in Eq. (2.1) the two binary operations are equal, i.e. $(\circ) = (\cdot)$, then the 3 tuple $(\alpha_1, \alpha_2, \alpha_3)$ is called an *autotopy* of binary groupoid $(G, \cdot)$. The set of all autotopies of a groupoid $(G, \cdot)$ is denoted by $\mathrm{Avt}(G, \cdot)$ and forms a group with respect to usual component-wise multiplications of autotopies. The last component of an autotopy of a groupoid (here α_3) is called a *quasiautomorphism.*

If σ is a parastrophe of a quasigroup $(Q, \cdot)$ and $T = (\alpha_1, \alpha_2, \alpha_3)$ is an isotopy, then we shall denote the action of parastrophe σ on the isotropic triplet T by $T^\sigma = (\alpha_{\sigma^{-1}1}, \alpha_{\sigma^{-1}2}, \alpha_{\sigma^{-1}3})$.

Definition 2.4. Let (Q, A) and (Q, B) be two quasigroups. Then (Q, B) is an *isostrophic image* of (Q, A), if there exists a collection of permutations $(\sigma, (\alpha_1, \alpha_2, \alpha_3))$ =(σ, T), where $\sigma \in S_3$, $T = (\alpha_1, \alpha_2, \alpha_3)$ and $\alpha_1, \alpha_2, \alpha_3$ are permutations of the set Q such that

$$B(x_1, x_2) = A(x_1, x_2)(\sigma, T) = \alpha_3^{-1} A(\alpha_1 x_{\sigma^{-1}1}, \alpha_2 x_{\sigma^{-1}2}) \tag{2.2}$$

for all $x_1, x_2 \in Q$.

In other words, an isostrophic image of a quasigroup is defined as an isotopic image of its parastrophe.

Let $(\sigma, T) = (\sigma, (\alpha_1, \alpha_2, \alpha_3))$ and $(\tau, S) = (\tau, (\beta_1, \beta_2, \beta_3))$ be two isostrophisms. Then their multiplication is defined as follows (successive multiplication of maps):

$$(\sigma, T)(\tau, S) = (\sigma\tau, T^\tau S). \tag{2.3}$$

The equality $B = A(\alpha\tau, T^\tau S)$ can be written in more details as follows (right record of maps):

$$B(x_1, x_2, x_3) = A(x_{1(\tau^{-1}\sigma^{-1})}\alpha_{1\tau^{-1}}\beta_1, x_{2(\tau^{-1}\sigma^{-1})}\alpha_{2\tau^{-1}}\beta_2, x_{3(\tau^{-1}\sigma^{-1})}\alpha_{3\tau^{-1}}\beta_3).$$

If we use the left record of maps, then

$$B(x_1, x_2, x_3) = A(\beta_1\alpha_{\tau^{-1}1}x_{\sigma^{-1}(\tau^{-1}1)}, \beta_2\alpha_{\tau^{-1}2}x_{\sigma^{-1}(\tau^{-1}2)}, \beta_3\alpha_{\tau^{-1}3}x_{\sigma^{-1}(\tau^{-1}3)}).$$

Also the inverse of the isostrophy (σ, T) is defined as

$$(\sigma, T)^{-1} = (\sigma^{-1}, (T^{-1})^{\sigma^{-1}}) = (\sigma^{-1}, (\alpha_{\sigma 1}^{-1}, \alpha_{\sigma 2}^{-1}, \alpha_{\sigma 2}^{-1})).$$

If in Definition 2.4, $(Q, B) = (Q, A)$ then (σ, T) is called an *autostrophy* of the quasigroup (Q, A). The set of all autostrophies of the quasigroup (Q, A) form a group under the operation defined in Eq. (2.3). We shall denote this set by $\mathrm{Aus}(Q, A)$.

Theorem 2.1 ([3]). *If quasigroup* $(Q, \circ)$ *is an isostrophic image of quasigroup* $(Q, \cdot)$ *with an isostrophy* T, *i.e.* $(Q, \circ) = (Q, \cdot)T$, *then* $\mathrm{Aus}(Q, \circ) = T^{-1}\mathrm{Aus}(Q, \cdot)T$.

Garrison [5] gave the concept of quasigroup nucleus which can be defined as follows.

Definition 2.5. The left, right and middle nuclei of a quasigroup $(Q, \cdot)$ are, respectively, defined as follows:

$$N_l = \{a \in Q \,|\, a \cdot (x \cdot y) = (a \cdot x) \cdot y, \forall\, x, y \in Q\},$$

$$N_r = \{a \in Q \,|\, (x \cdot y) \cdot a = x \cdot (y \cdot a), \forall\, x, y \in Q\}$$

and

$$N_m = \{a \in Q \,|\, (x \cdot a) \cdot y = x \cdot (a \cdot y), \forall\, x, y \in Q\}.$$

The nucleus of the quasigroup $(Q, \cdot)$ is the intersection of the above defined three, i.e. $N = N_l \cap N_r \cap N_m$.

The goodness of Garrison's quasigroup nuclei is that if the set N_l (N_r, N_m) is non-empty then it forms a subgroup of the quasigroup $(Q, \cdot)$. But there is a weakness that if a quasigroup $(Q, \cdot)$ has a non-empty left or right or middle nucleus then the quasigroup is a left loop or a right loop or a loop, respectively [5, 8].

Definition 2.6 ([11]). The set of all autotopisms of the form $(\alpha, \varepsilon, \gamma)$ of a quasigroup $(Q, \cdot)$ is called the left A-nucleus (left autotopy nucleus) of quasigroup $(Q, \cdot)$, where ε is the identity mapping.

Similarly, the set of autotopisms of the forms $(\varepsilon, \beta, \gamma)$ and $(\alpha, \beta, \varepsilon)$ form the right and middle A-nucleus of the quasigroup $(Q, \cdot)$, respectively. Here α, β, γ are permutations of the set Q. We shall, respectively, denote these three sets of autotopisms by N_l^A, N_r^A and N_m^A and their sets of all ith components by ${}_iN_l^A$, ${}_iN_r^A$ and ${}_iN_m^A$, respectively.

Table 2 shows connections between components of the A-nuclei of a quasigroup $(Q, \cdot)$ and its isostrophic images of the form $(Q, \circ) = (Q, \cdot)(\sigma, (\alpha, \beta, \gamma))$, where $\sigma \in S_3$, $\alpha, \beta, \gamma \in S_Q$ [11].

Theorem 2.2 ([12]). *If* $(Q, \cdot)$ *is a loop with identity element* 1 *then*:

(1) $(\alpha, \varepsilon, \gamma) \in \mathrm{Avt}(Q, \cdot) \Leftrightarrow \alpha = \gamma = L_{\alpha 1}$ *and* $\alpha 1 \in N_l$,

Table 2. Connections between components of A-nuclei by isostrophy.

	(ε, T)	$((12), T)$	$((13), T)$	$((23), T)$	$((132), T)$	$((123), T)$
${}_1N_l^\circ$	$\alpha^{-1}\, {}_1N_l^\cdot \alpha$	$\alpha^{-1}\, {}_2N_r^\cdot \alpha$	$\alpha^{-1}\, {}_3N_l^\cdot \alpha$	$\alpha^{-1}\, {}_1N_m^\cdot \alpha$	$\alpha^{-1}\, {}_2N_m^\cdot \alpha$	$\alpha^{-1}\, {}_3N_r^\cdot \alpha$
${}_3N_l^\circ$	$\gamma^{-1}\, {}_3N_l^\cdot \gamma$	$\gamma^{-1}\, {}_3N_r^\cdot \gamma$	$\gamma^{-1}\, {}_1N_l^\cdot \gamma$	$\gamma^{-1}\, {}_2N_m^\cdot \gamma$	$\gamma^{-1}\, {}_1N_m^\cdot \gamma$	$\gamma^{-1}\, {}_2N_r^\cdot \gamma$
${}_2N_r^\circ$	$\beta^{-1}\, {}_2N_r^\cdot \beta$	$\beta^{-1}\, {}_1N_l^\cdot \beta$	$\beta^{-1}\, {}_2N_m^\cdot \beta$	$\beta^{-1}\, {}_3N_r^\cdot \beta$	$\beta^{-1}\, {}_3N_l^\cdot \beta$	$\beta^{-1}\, {}_1N_m^\cdot \beta$
${}_3N_r^\circ$	$\gamma^{-1}\, {}_3N_r^\cdot \gamma$	$\gamma^{-1}\, {}_3N_l^\cdot \gamma$	$\gamma^{-1}\, {}_1N_m^\cdot \gamma$	$\gamma^{-1}\, {}_2N_r^\cdot \gamma$	$\gamma^{-1}\, {}_1N_l^\cdot \gamma$	$\gamma^{-1}\, {}_2N_m^\cdot \gamma$
${}_1N_m^\circ$	$\alpha^{-1}\, {}_1N_m^\cdot \alpha$	$\alpha^{-1}\, {}_2N_m^\cdot \alpha$	$\alpha^{-1}\, {}_3N_r^\cdot \alpha$	$\alpha^{-1}\, {}_1N_l^\cdot \alpha$	$\alpha^{-1}\, {}_2N_r^\cdot \alpha$	$\alpha^{-1}\, {}_3N_l^\cdot \alpha$
${}_2N_m^\circ$	$\beta^{-1}\, {}_2N_m^\cdot \beta$	$\beta^{-1}\, {}_1N_m^\cdot \beta$	$\beta^{-1}\, {}_2N_r^\cdot \beta$	$\beta^{-1}\, {}_3N_l^\cdot \beta$	$\beta^{-1}\, {}_3N_r^\cdot \beta$	$\beta^{-1}\, {}_1N_l^\cdot \beta$

(2) $(\varepsilon, \beta, \gamma) \in \mathrm{Avt}(Q, \cdot) \Leftrightarrow \beta = \gamma = R_{\beta 1}$ *and* $\beta 1 \in N_r$,
(3) $(\alpha, \beta, \varepsilon) \in \mathrm{Avt}(Q, \cdot) \Leftrightarrow \alpha = R_{\beta 1}, \beta = L_{\beta 1}^{-1}$ *and* $\beta 1 \in N_m$.

3. σ-A-Nuclei and A-Nuclei of a quasigroup

The left, right and middle A-nuclei of a quasigroup can be generalized to left, right and middle σ-A-nuclei, for $\sigma \in S_3$. A definition of σ-A-nuclei is given in [12]. We shall redefine left, right and middle σ-A-nuclei of a quasigroup as follows.

Definition 3.1. The set of all autostrophies of a quasigroup $(Q, \cdot)$ of the form $(\sigma, (\alpha, \varepsilon, \gamma))$, $(\sigma, (\varepsilon, \beta, \gamma))$ and $(\sigma, (\alpha, \beta, \varepsilon))$ are, respectively, called the left, right and middle σ-A-nuclei of the quasigroup $(Q, \cdot)$, where α, β, γ are permutations of the set Q and $\sigma \in S_3$.

For $\sigma = \varepsilon$ in the above definition we get left (right, middle) ε-A-nucleus of the quasigroup which can be considered as left (right, middle) A-nucleus of the quasigroup.

We shall denote the left, right and middle σ-A-nuclei by ${}^{\sigma}N_l^A$, ${}^{\sigma}N_r^A$ and ${}^{\sigma}N_m^A$, respectively.

Example 3.1. Let $Q = \{1, 2, 3\}$ and a quasigroup $(Q, \cdot)$ be as shown below:

$\cdot$	1	2	3
1	2	3	1
2	1	2	3
3	3	1	2

The left, right and middle (12)-A-nuclei of the quasigroup $(Q, \cdot)$ are, respectively, as follows:

$$^{(12)}N_l^A = \{((12), (\varepsilon, \varepsilon, (13))), ((12), ((123), \varepsilon, (12))), ((12), ((132), \varepsilon, (23)))\},$$

$$^{(12)}N_r^A = \{((12), (\varepsilon, \varepsilon, (13))), ((12), (\varepsilon, (132), (12))), ((12), (\varepsilon, (123), (23)))\},$$

$$^{(12)}N_m^A = \{((12), ((12), (12), \varepsilon)), ((12), ((13), (13), \varepsilon)), ((12), ((23), (23), \varepsilon))\}.$$

In the following three theorems, we give the explicit description of left, right and middle σ-A-nuclei of a quasigroup.

Theorem 3.1. *In any quasigroup* $(Q, \cdot)$ *we have*

$$N_l^A = {}^{\varepsilon}N_l^A = \{(\varepsilon, (\alpha, \varepsilon, R_a \alpha R_a^{-1})) \mid \forall\, a \in Q\} = \{(\varepsilon, (R_a^{-1} \gamma R_a, \varepsilon, \gamma)) \mid \forall\, a \in Q\},$$

$$^{(12)}N_l^A = \{((12), (\alpha, \varepsilon, R_a \alpha L_a^{-1})) \mid \forall\, a \in Q\} = \{((12), (R_a^{-1} \gamma L_a, \varepsilon, \gamma)) \mid \forall\, a \in Q\},$$

$$^{(13)}N_l^A = \{((13), (\alpha, \varepsilon, R_a \alpha R_a)) \mid \forall\, a \in Q\} = \{((13), (R_a^{-1} \gamma R_a^{-1}, \varepsilon, \gamma)) \mid \forall\, a \in Q\},$$

$$^{(23)}N_l^A = \{((23),(\alpha,\varepsilon,L_{\alpha a}L_a))\,|\,\forall\, a\in Q\} = \{((23),(P_{\gamma a}^{-1}R_a,\varepsilon,\gamma))\,|\,\forall\, a\in Q\},$$

$$^{(123)}N_l^A = \{((123),(\alpha,\varepsilon,L_{\alpha a}R_a))\,|\,\forall\, a\in Q\} = \{((123),(P_{\gamma a}^{-1}L_a,\varepsilon,\gamma))\,|\,\forall\, a\in Q\},$$

$$^{(132)}N_l^A = \{((132),(\alpha,\varepsilon,R_a\alpha L_a))\,|\,\forall\, a\in Q\} = \{((132),(R_a^{-1}\gamma L_a^{-1},\varepsilon,\gamma))\,|\,\forall\, a\in Q\}.$$

Proof. If $(\sigma,(\alpha_1,\alpha_2,\alpha_3))$ is an autostrophism of a quasigroup (Q,A), then from definition of autostrophy we have

$$A(x_1,x_2,x_3) = A(\alpha_1 x_{\sigma^{-1}1},\alpha_2 x_{\sigma^{-1}2},\alpha_3 x_{\sigma^{-1}3}) \tag{3.1}$$

for all $x_1,x_2,x_3\in Q$.

Case 1. If $\sigma=\varepsilon$. From Eq. (3.1), for $\sigma=\varepsilon$ and $\alpha_1=\alpha,\alpha_2=\varepsilon,\alpha_3=\gamma$ we get $(\varepsilon,(\alpha,\varepsilon,\gamma)\in {}^{\varepsilon}N_l^A \Leftrightarrow \alpha x\cdot y = \gamma(x\cdot y) \Leftrightarrow R_y\alpha x = \gamma R_y x \Leftrightarrow \gamma = R_y\alpha R_y^{-1}$ or $\alpha = R_y^{-1}\gamma R_y\ \forall\, y\in Q$.

Case 2. If $\sigma=(12)$. From Eq. (3.1), for $\sigma=(12)$ and $\alpha_1=\alpha,\alpha_2=\varepsilon,\alpha_3=\gamma$ we get $(12)(\alpha,\varepsilon,\gamma)\in {}^{(12)}N_l^A \Leftrightarrow \alpha y\cdot x = \gamma(x\cdot y)\ R_x\alpha y = \gamma(L_x y) \Leftrightarrow \gamma = R_x\alpha L_x^{-1}$ or $\alpha = R_x^{-1}\gamma L_x\ \forall\, x\in Q$.

Case 3. If $\sigma=(13)$. From Eq. (3.1), for $\sigma=(13)$ and $\alpha_1=\alpha,\alpha_2=\varepsilon,\alpha_3=\gamma$ we get $(13)(\alpha,\varepsilon,\gamma)\in {}^{(13)}N_l^A \Leftrightarrow \alpha(x\cdot y)\cdot y = \gamma x \Leftrightarrow \gamma = R_y\alpha R_y$ or $\alpha = R_y^{-1}\gamma R_y^{-1}$ $\forall\, y\in Q$.

Case 4. If $\sigma=(23)$. From Eq. (3.1), for $\sigma=(23)$ and $\alpha_1=\alpha,\alpha_2=\varepsilon,\alpha_3=\gamma$ we get $(23)(\alpha,\varepsilon,\gamma)\in {}^{(23)}N_l^A \Leftrightarrow \alpha(x\cdot y)\cdot y = \gamma x \Leftrightarrow \alpha x\cdot(x\cdot y) = \gamma y \Leftrightarrow \gamma = L_{\alpha x}L_x$ or $\alpha x = \gamma y/R_y x \Leftrightarrow \alpha = P_{\gamma y}^{-1}R_y\ \forall\, y\in Q$.

Case 5. If $\sigma=(123)$. From Eq. (3.1), for $\sigma=(123)$ and $\alpha_1=\alpha,\alpha_2=\varepsilon,\alpha_3=\gamma$ we get $(123)(\alpha,\varepsilon,\gamma)\in {}^{(123)}N_l^A \Leftrightarrow \alpha y\cdot(x\cdot y) = \gamma x$ or $\alpha y = \gamma x/L_x y \Leftrightarrow \gamma = L_{\alpha y}R_y$ or $\alpha = P_{\gamma x}^{-1}L_x\ \forall\, x\in Q$.

Case 6. If $\sigma=(132)$. From Eq. (3.1), for $\sigma=(132)$ and $\alpha_1=\alpha,\alpha_2=\varepsilon,\alpha_3=\gamma$ we get $(132)(\alpha,\varepsilon,\gamma)\in {}^{(132)}N_l^A \Leftrightarrow \alpha(x\cdot y)\cdot x = \gamma y \Leftrightarrow R_x\alpha L_x y = \gamma y \Leftrightarrow \gamma = R_x\alpha L_x$ or $\alpha = R_x^{-1}\gamma L_x^{-1}\ \forall\, x\in Q$. □

Theorem 3.2. *In any quasigroup $(Q,\cdot)$ we have*

$$N_r^A = {}^{\varepsilon}N_r^A = \{(\varepsilon,(\varepsilon,\beta,L_a\beta L_a^{-1}))\,|\,\forall\, a\in Q\} = \{(\varepsilon,(\varepsilon,L_a^{-1}\gamma L_a,\gamma))\,|\,\forall\, a\in Q\},$$

$$^{(12)}N_r^A = \{((12),(\varepsilon,\beta,L_a\beta R_a^{-1}))\,|\,\forall\, a\in Q\} = \{((12),(\varepsilon,L_a^{-1}\gamma R_a,\gamma))\,|\,\forall\, a\in Q\},$$

$$^{(13)}N_r^A = \{((13),(\varepsilon,\beta,R_{\beta a}R_a))\,|\,\forall\, a\in Q\} = \{((13),(\varepsilon,P_{\gamma a}L_a,\gamma))\,|\,\forall\, a\in Q\},$$

$$^{(23)}N_r^A = \{((23),(\varepsilon,\beta,L_a\beta L_a))\,|\,\forall\, a\in Q\} = \{((23),(\varepsilon,P_{\gamma a}R_a^{-1},\gamma))\,|\,\forall\, a\in Q\},$$

$$^{(123)}N_r^A = \{((123),(\varepsilon,\beta,L_a\beta R_a))\,|\,\forall\, a\in Q\} = \{((123),(\varepsilon,P_{\gamma a}L_a^{-1},\gamma))\,|\,\forall\, a\in Q\},$$

$$^{(132)}N_r^A = \{((132),(\varepsilon,\beta,L_{\beta a}L_a))\,|\,\forall\, a\in Q\} = \{((132),(\varepsilon,P_{\gamma a}R_a,\gamma))\,|\,\forall\, a\in Q\}.$$

Proof. This can be proved on similar lines as in Theorem 3.1 with $\alpha_1=\varepsilon,\alpha_2=\beta$ and $\alpha_3=\gamma$. □

Theorem 3.3. *In any quasigroup $(Q,\cdot)$ we have*

$$N_m^A = {}^{\varepsilon}N_m^A = \{(\varepsilon,(\alpha,P_a\alpha P_a^{-1},\varepsilon))\,|\,\forall\, a\in Q\} = \{(\varepsilon,(P_a^{-1}\beta P_a,\beta,\varepsilon))\,|\,\forall\, a\in Q\},$$
$${}^{(12)}N_m^A = \{((12),(\alpha,P_a^{-1}\alpha P_a^{-1},\varepsilon))\,|\,\forall\, a\in Q\} = \{((12),(P_a\beta P_a,\beta,\varepsilon))\,|\,\forall\, a\in Q\},$$
$${}^{(13)}N_m^A = \{((13),(\alpha,P_a\alpha L_a,\varepsilon))\,|\,\forall\, a\in Q\} = \{((13),(R_{\beta a}^{-1}R_a^{-1},\beta,\varepsilon))\,|\,\forall\, a\in Q\},$$
$${}^{(23)}N_m^A = \{((23),(\alpha,L_{\alpha a}^{-1}L_a^{-1},\varepsilon))\,|\,\forall\, a\in Q\} = \{((23),(P_a\beta R_a,\beta,\varepsilon))\,|\,\forall\, a\in Q\},$$
$${}^{(123)}N_m^A = \{((123),(\alpha,L_{\alpha a}^{-1}R_a^{-1},\varepsilon))\,|\,\forall\, a\in Q\} = \{((123),(P_a^{-1}\beta L_a,\beta,\varepsilon))\,|\,\forall\, a\in Q\},$$
$${}^{(132)}N_m^A = \{((132),(\alpha,P_a\alpha R_a,\varepsilon))\,|\,\forall\, a\in Q\} = \{((132),(R_{\beta a}^{-1}L_a^{-1},\beta,\varepsilon))\,|\,\forall\, a\in Q\}.$$

Proof. This can be proved on similar lines as in Theorem 3.1 with $\alpha_1=\alpha,\alpha_2=\beta$ and $\alpha_3=\varepsilon$. □

In the following theorem, we derive connections between components of σ-A-nuclei and local identity elements of a quasigroup.

Lemma 3.1. *Let $(Q,\cdot)$ be a quasigroup.*

(1) (a) *If* $(\varepsilon,(\alpha,\varepsilon,\gamma))\in \mathrm{Aus}(Q,\cdot)$ *then* $\alpha x\cdot e(x)=\gamma x$, $\alpha f(x)\cdot x=\gamma x$ *and* $\alpha x\cdot x=\gamma s(x)$ *for all* $x\in Q$.
   (b) *If* $((12),(\alpha,\varepsilon,\gamma))\in \mathrm{Aus}(Q,\cdot)$ *then* $\alpha e(x)\cdot x=\gamma x$, $\alpha x\cdot f(x)=\gamma x$ *and* $\alpha x\cdot x=\gamma s(x)$ *for all* $x\in Q$.
   (c) *If* $((13),(\alpha,\varepsilon,\gamma))\in \mathrm{Aus}(Q,\cdot)$ *then* $\alpha x\cdot e(x)=\gamma x$, $\alpha x\cdot x=\gamma f(x)$ *and* $\alpha s(x)\cdot x=\gamma x$ *for all* $x\in Q$.
   (d) *If* $((23),(\alpha,\varepsilon,\gamma))\in \mathrm{Aus}(Q,\cdot)$ *then* $\alpha x\cdot x=\gamma e(x)$, $\alpha f(x)\cdot x=\gamma x$ *and* $\alpha x\cdot s(x)=\gamma x$ *for all* $x\in Q$.
   (e) *If* $((123),(\alpha,\varepsilon,\gamma))\in \mathrm{Aus}(Q,\cdot)$ *then* $\alpha x\cdot x=\gamma e(x)$, $\alpha x\cdot f(x)=\gamma x$ *and* $\alpha s(x)\cdot x=\gamma x$ *for all* $x\in Q$.
   (f) *If* $((132),(\alpha,\varepsilon,\gamma))\in \mathrm{Aus}(Q,\cdot)$ *then* $\alpha e(x)\cdot x=\gamma x$, $\alpha x\cdot x=\gamma f(x)$ *and* $\alpha x\cdot s(x)=\gamma x$ *for all* $x\in Q$.

(2) (a) *If* $((\varepsilon),(\varepsilon,\beta,\gamma))\in \mathrm{Aus}(Q,\cdot)$ *then* $x\cdot\beta e(x)=\gamma x$, $f(x)\cdot\beta x=\gamma x$ *and* $x\cdot\beta x=\gamma s(x)$ *for all* $x\in Q$.
   (b) *If* $((12),(\varepsilon,\beta,\gamma))\in \mathrm{Aus}(Q,\cdot)$ *then* $e(x)\cdot\beta x=\gamma x$, $x\cdot\beta f(x)=\gamma x$ *and* $x\cdot\beta x=\gamma s(x)$ *for all* $x\in Q$.
   (c) *If* $((13),(\varepsilon,\beta,\gamma))\in \mathrm{Aus}(Q,\cdot)$ *then* $x\cdot\beta e(x)=\gamma x$, $x\cdot\beta x=\gamma f(x)$ *and* $s(x)\cdot\beta x=\gamma x$ *for all* $x\in Q$.
   (d) *If* $((23),(\varepsilon,\beta,\gamma))\in \mathrm{Aus}(Q,\cdot)$ *then* $x\cdot\beta x=\gamma e(x)$, $f(x)\cdot\beta x=\gamma x$ *and* $x\cdot\beta s(x)=\gamma x$ *for all* $x\in Q$.
   (e) *If* $((123),(\varepsilon,\beta,\gamma))\in \mathrm{Aus}(Q,\cdot)$ *then* $x\cdot\beta x=\gamma e(x)$, $x\cdot\beta f(x)=\gamma x$ *and* $s(x)\cdot\beta x=\gamma x$ *for all* $x\in Q$.
   (f) *If* $((132),(\varepsilon,\beta,\gamma))\in \mathrm{Aus}(Q,\cdot)$ *then* $e(x)\cdot\beta x=\gamma x$, $x\cdot\beta x=\gamma f(x)$ *and* $x\cdot\beta s(x)=\gamma x$ *for all* $x\in Q$.

(3) (a) *If* $(\varepsilon, (\alpha, \varepsilon, \gamma)) \in \mathrm{Aus}(Q, \cdot)$ *then* $\alpha x \cdot \beta e(x) = x$, $\alpha f(x) \cdot \beta x = x$ *and* $\alpha x \cdot \beta x = s(x)$ *for all* $x \in Q$.
(b) *If* $((12), (\alpha, \varepsilon, \gamma)) \in \mathrm{Aus}(Q, \cdot)$ *then* $\alpha e(x) \cdot \beta x = x$, $\alpha x \cdot \beta f(x) = x$ *and* $\alpha x \cdot \beta x = s(x)$ *for all* $x \in Q$.
(c) *If* $((13), (\alpha, \varepsilon, \gamma)) \in \mathrm{Aus}(Q, \cdot)$ *then* $\alpha x \cdot \beta e(x) = x$, $\alpha x \cdot \beta x = f(x)$ *and* $\alpha s(x) \cdot \beta x = x$ *for all* $x \in Q$.
(d) *If* $((23), (\alpha, \varepsilon, \gamma)) \in \mathrm{Aus}(Q, \cdot)$ *then* $\alpha x \cdot \beta x = e(x)$, $\alpha f(x) \cdot \beta x = x$ *and* $\alpha x \cdot \beta s(x) = x$ *for all* $x \in Q$.
(e) *If* $((123), (\alpha, \varepsilon, \gamma)) \in \mathrm{Aus}(Q, \cdot)$ *then* $\alpha x \cdot \beta x = e(x)$, $\alpha x \cdot \beta f(x) = x$ *and* $\alpha s(x) \cdot \beta x = x$ *for all* $x \in Q$.
(f) *If* $((132), (\alpha, \varepsilon, \gamma)) \in \mathrm{Aus}(Q, \cdot)$ *then* $\alpha e(x) \cdot \beta x = x$, $\alpha x \cdot \beta x = f(x)$ *and* $\alpha x \cdot \beta s(x) = x$ *for all* $x \in Q$.

Proof. Case 1. If $(\varepsilon, (\alpha, \varepsilon, \gamma) \in \mathrm{Aus}(Q, \cdot)$ then from Eq. (3.1), for $\sigma = \varepsilon$ and $\alpha_1 = \alpha, \alpha_2 = \varepsilon, \alpha_3 = \gamma$ we have $\alpha x \cdot y = \gamma(x \cdot y)$. Putting $y = e(x)$ we get $\alpha x \cdot e(x) = \gamma x \; \forall\, x \in Q$. Putting $x = f(y)$ we get $\alpha f(x) \cdot x = \gamma x \; \forall\, x \in Q$. Putting $y = x$ we get $\alpha x \cdot x = \gamma s(x) \; \forall\, x \in Q$.

If $((12), (\alpha, \varepsilon, \gamma)) \in \mathrm{Aus}(Q, \cdot)$ then from Eq. (3.1), for $\sigma = (12)$ and $\alpha_1 = \alpha, \alpha_2 = \varepsilon, \alpha_3 = \gamma$ we have $\alpha y \cdot x = \gamma(x \cdot y)$. Putting $y = e(x)$ we get $\alpha e(x) \cdot x = \gamma x \; \forall\, x \in Q$. Putting $x = f(y)$ we get $\alpha x \cdot f(x) = \gamma x \; \forall\, x \in Q$. Putting $y = x$ we get $\alpha x \cdot x = \gamma s(x)$ $\forall\, x \in Q$.

If $((13), (\alpha, \varepsilon, \gamma)) \in \mathrm{Aus}(Q, \cdot)$ then from Eq. (3.1), for $\sigma = (13)$ and $\alpha_1 = \alpha, \alpha_2 = \varepsilon, \alpha_3 = \gamma$ we get $\alpha(x \cdot y) \cdot y = \gamma x$. Putting $y = e(x)$ we get $\alpha x \cdot e(x) = \gamma x \; \forall\, x \in Q$. Putting $x = f(y)$ we get $\alpha x \cdot x = \gamma f(x) \; \forall\, x \in Q$. Putting $y = x$ we get $\alpha s(x) \cdot x = \gamma x$ $\forall\, x \in Q$.

If $((23), (\alpha, \varepsilon, \gamma)) \in \mathrm{Aus}(Q, \cdot)$ then from Eq. (3.1), for $\sigma = (23)$ and $\alpha_1 = \alpha, \alpha_2 = \varepsilon, \alpha_3 = \gamma$ we get $\alpha x \cdot (x \cdot y) = \gamma y$. Putting $y = e(x)$ we get $\alpha x \cdot x = \gamma e(x) \; \forall\, x \in Q$. Putting $x = f(y)$ we get $\alpha f(x) \cdot x = \gamma x \; \forall\, x \in Q$. Putting $y = x$ we get $\alpha x \cdot s(x) = \gamma x$ $\forall\, x \in Q$.

If $((123), (\alpha, \varepsilon, \gamma)) \in \mathrm{Aus}(Q, \cdot)$ then from Eq. (3.1), for $\sigma = (123)$ and $\alpha_1 = \alpha, \alpha_2 = \varepsilon, \alpha_3 = \gamma$ we get $\alpha(x \cdot y) \cdot x = \gamma y$. Putting $y = e(x)$ we get $\alpha x \cdot x = \gamma e(x)$ $\forall\, x \in Q$. Putting $x = f(y)$ we get $\alpha x \cdot f(x) = \gamma x \; \forall\, x \in Q$. Putting $y = x$ we get $\alpha s(x) \cdot x = \gamma x \; \forall\, x \in Q$.

If $((132), (\alpha, \varepsilon, \gamma)) \in \mathrm{Aus}(Q, \cdot)$ then from Eq. (3.1), for $\sigma = (132)$ and $\alpha_1 = \alpha, \alpha_2 = \varepsilon, \alpha_3 = \gamma$ we get $\alpha y \cdot (x \cdot y) = \gamma x$. Putting $y = e(x)$ we get $\alpha e(x) \cdot x = \gamma x$ $\forall\, x \in Q$. Putting $x = f(y)$ we get $\alpha x \cdot x = \gamma f(x) \; \forall\, x \in Q$. Putting $y = x$ we get $\alpha x \cdot s(x) = \gamma x \; \forall\, x \in Q$.

Other cases can be proved similarly. □

Corollary 3.1. (1) *If* $(Q, \cdot)$ *is an idempotent quasigroup and* $(\sigma, (\alpha, \varepsilon, \gamma)) \in \mathrm{Aus}(Q, \cdot)$, *then* $\alpha x = P_{\gamma x}^{-1} x \; \forall\, x \in Q$ *and* $\sigma \in S_3$.
(2) *If* $(Q, \cdot)$ *is an idempotent quasigroup and* $(\sigma, (\varepsilon, \beta, \gamma)) \in \mathrm{Aus}(Q, \cdot)$, *then* $\beta x = P_{\gamma x} x \; \forall\, x \in Q$ *and* $\sigma \in S_3$.

(3) *If* $(Q,\cdot)$ *is an idempotent quasigroup and* $(\sigma,(\alpha,\beta,\varepsilon))\in \mathrm{Aus}(Q,\cdot)$, *then* $\beta x = P_x\alpha x\ \forall\, x\in Q$ *and* $\sigma\in S_3$.

Proof. In an idempotent quasigroup, the maps e, f and s are equal to identity permutation of the set Q. From case (1) of Lemma 3.1 we get $\alpha x\cdot x=\gamma x$ which implies $\alpha x=\gamma x/x$. Hence, $\alpha x = P^{-1}_{\gamma x}x\ \forall\, x\in Q$ and $\sigma\in S_3$. From case (2) of Lemma 3.1 we get $x\cdot\beta x=\gamma x$ which implies $\beta x = x\backslash\gamma x$. Hence, $\beta x = P_{\gamma x}x$ $\forall\, x\in Q$ and $\sigma\in S_3$. From case (3) of Lemma 3.1 we get $\alpha x\cdot\beta x = x$ which implies $\alpha x\backslash x=\beta x$. Hence, $\beta x = P_x\alpha x\ \forall\, x\in Q$ and $\sigma\in S_3$. □

Remark 3.1. In a left loop $f(x)=1\ \forall\, x\in Q$, in a right loop $e(x)=1\ \forall\, x\in Q$ and in a unipotent quasigroup $s(x)=1\ \forall\, x\in Q$, where 1 is a fixed element of the set Q.

Corollary 3.2. *Let* $(Q,\cdot)$ *be a left loop with left identity element* $f=1$ *and* $\sigma\in S_3$.

(1) *If* $(\sigma,(\alpha,\varepsilon,\gamma))\in\mathrm{Aus}(Q,\cdot)$ *then*:
 (a) $\gamma = L_{\alpha 1}$ *for* $\sigma=\varepsilon,(23)$,
 (b) $\gamma = R_1\alpha$ *for* $\sigma=(12),(123)$,
 (c) $\alpha = P^{-1}_{\gamma 1}$ *for* $\sigma=(13),(132)$.

(2) *If* $(\sigma,(\varepsilon,\beta,\gamma))\in\mathrm{Aus}(Q,\cdot)$ *then*:
 (a) $\beta=\gamma$ *for* $\sigma=\varepsilon,(23)$,
 (b) $\gamma = R_{\beta 1}$ *for* $\sigma=(12),(123)$,
 (c) $\beta = P_{\gamma 1}$ *for* $\sigma=(13),(132)$.

(3) *If* $(\sigma,(\alpha,\beta,\varepsilon))\in\mathrm{Aus}(Q,\cdot)$ *then*:
 (a) $\beta = L^{-1}_{\alpha 1}$ *for* $\sigma=\varepsilon,(23)$,
 (b) $\alpha = R^{-1}_{\beta 1}$ *for* $\sigma=(12),(123)$,
 (c) $\beta = P_1\alpha$ *for* $\sigma=(13),(132)$.

Proof. Case 1. Putting $f(x)=1\ \forall\, x\in Q$ in the case (1) of Lemma 3.1 we obtain the following: for $\sigma=\varepsilon,(23)$ we get $\alpha 1\cdot x=\gamma x$, i.e. $\gamma x = L_{\alpha 1}x\ \forall\, x\in Q$; for $\sigma=(12),(123)$ we get $\alpha x\cdot 1=\gamma x$, i.e. $\gamma x = R_1\alpha x\ \forall\, x\in Q$; for $\sigma=(13),(132)$ we get $\alpha x\cdot x=\gamma 1$ which implies $\alpha x=\gamma 1/x$, i.e. $\alpha x = P^{-1}_{\gamma 1}x\ \forall\, x\in Q$.

Other cases can be proved similarly. □

Corollary 3.3. *Let* $(Q,\cdot)$ *be a right loop with right identity element* $e=1$ *and* $\sigma\in S_3$.

(1) *If* $(\sigma,(\alpha,\varepsilon,\gamma))\in\mathrm{Aus}(Q,\cdot)$ *then*:
 (a) $\alpha=\gamma$ *for* $\sigma=\varepsilon,(13)$,
 (b) $\gamma = L_{\alpha 1}$ *for* $\sigma=(12),(132)$,
 (c) $\alpha = P^{-1}_{\gamma 1}$ *for* $\sigma=(23),(123)$.

(2) *If* $(\sigma,(\varepsilon,\beta,\gamma))\in\mathrm{Aus}(Q,\cdot)$ *then*:
 (a) $\gamma = R_{\beta 1}$ *for* $\sigma=\varepsilon,(13)$,

  (b) $\gamma = L_1\beta$ *for* $\sigma = (12), (132)$,
  (c) $\beta = P_{\gamma 1}$ *for* $\sigma = (23), (123)$.

(3) *If* $(\sigma, (\alpha, \beta, \varepsilon)) \in \mathrm{Aus}(Q, \cdot)$ *then*:

  (a) $\alpha = R_{\beta 1}^{-1}$ *for* $\sigma = \varepsilon, (13)$,
  (b) $\beta = L_{\alpha 1}^{-1}$ *for* $\sigma = (12), (132)$,
  (b) $\alpha = P_1\alpha$ *for* $\sigma = (23), (123)$,

Proof. Case 2. Putting $e(x) = 1\ \forall\, x \in Q$ in the case (2) of Lemma 3.1 we obtain the following: for $\sigma = \varepsilon, (13)$, $x{\cdot}\beta 1 = \gamma x$, i.e. $\gamma x = R_{\beta 1}x\ \forall\, x \in Q$; for $\sigma = (12), (132)$, $1.\beta x = \gamma x$, i.e. $\gamma x = L_1\beta x\ \forall\, x \in Q$; for $\sigma = (23), (123)$, $x \cdot \beta x = \gamma 1$ which implies $\beta x = x\backslash\gamma x$, i.e. $\beta x = P_{\gamma 1}x\ \forall\, x \in Q$.

Other cases can be proved similarly. □

Corollary 3.4. *Let* $(Q, \cdot)$ *be a unipotent quasigroup with middle identity element* $s = 1$ *and* $\sigma \in S_3$.

(1) *If* $(\sigma, (\alpha, \varepsilon, \gamma)) \in \mathrm{Aus}(Q, \cdot)$ *then*:

  (a) $\alpha = P_{\gamma 1}^{-1}$ *for* $\sigma = \varepsilon, (12)$,
  (b) $\gamma = L_{\alpha 1}$ *for* $\sigma = (13), (123)$,
  (c) $\gamma = R_1\alpha$ *for* $\sigma = (23), (132)$.

(2) *If* $(\sigma, (\varepsilon, \beta, \gamma)) \in \mathrm{Aus}(Q, \cdot)$ *then*:

  (a) $\alpha = P_{\gamma 1}$ *for* $\sigma = \varepsilon, (12)$,
  (b) $\gamma = L_1\beta$ *for* $\sigma = (13), (123)$,
  (c) $\gamma = R_{\beta 1}$ *for* $\sigma = (23), (132)$.

(3) *If* $(\sigma, (\alpha, \beta, \varepsilon)) \in \mathrm{Aus}(Q, \cdot)$ *then*:

  (a) $\alpha = \beta$ *for* $\sigma = \varepsilon, (12)$,
  (b) $\beta = L_{\alpha 1}^{-1}$ *for* $\sigma = (13), (123)$,
  (c) $\alpha = R_{\beta 1}^{-1}$ *for* $\sigma = (23), (132)$.

Proof. Case 3. Putting $s(x) = 1\ \forall\, x \in Q$ in the case (3) of Lemma 3.1 we obtain the following: for $\sigma = \varepsilon, (12)$, $\alpha x \cdot \beta x = 1$ which implies $\beta x = \alpha x\backslash 1$, i.e. $\beta x = P_1\alpha x\ \forall\, x \in Q$. But $P_1 = \varepsilon$ in a unipotent quasigroup. Hence, $\beta = \alpha$. Now for $\sigma = (13), (123)$, $\alpha 1.\beta x = x$, i.e. $\beta x = L_{\alpha 1}^{-1}x\ \forall\, x \in Q$ and for $\sigma = (23), (132)$, $\alpha x \cdot \beta 1 = x$, i.e. $\alpha x = R_{\beta 1}^{-1}x\ \forall\, x \in Q$.

Other cases can be proved similarly. □

In the following three theorems, we explicitly derive the structure of σ-A-nuclei of a loop, a unipotent left loop and a unipotent right loop in terms of translation maps. This would make the study of σ-A-nuclei of a loop, a unipotent left loop and a unipotent right loop much easier. As well as this will reduce the time required in computation of σ-A-nuclei of the mentioned quasigroups.

Theorem 3.4. *Let $(Q, \cdot)$ be a loop with identity element 1 and $\sigma \in S_3$.*

(1) *If $(\sigma, (\alpha, \varepsilon, \gamma)) \in \mathrm{Aus}(Q, \cdot)$ then*:

  (a) $\alpha = \gamma = L_{\alpha 1}$ *for* $\sigma = \varepsilon, (12)$,
  (b) $\alpha = \gamma = P_{\alpha 1}^{-1}$ *for* $\sigma = (13), (123)$,
  (c) $\alpha = P_{\alpha 1}^{-1}$, $\gamma = L_{\alpha 1}$ *for* $\sigma = (23), (132)$.

(2) *If $((\sigma, (\varepsilon, \beta, \gamma)) \in \mathrm{Aus}(Q, \cdot)$ then*:

  (a) $\beta = \gamma = R_{\beta 1}$ *for* $\sigma = \varepsilon, (12)$,
  (b) $\beta = P_{\beta 1}$, $\gamma = R_{\beta 1}$ *for* $\sigma = (13), (123)$,
  (c) $\beta = \gamma = P_{\beta 1}$ *for* $\sigma = (23), (132)$.

(3) *If $(\sigma, (\alpha, \beta, \varepsilon)) \in \mathrm{Aus}(Q, \cdot)$ then*:

  (a) $\alpha = R_{\beta 1}^{-1}$, $\beta = L_{\alpha 1}^{-1}$ *for* $\sigma = \varepsilon, (12)$,
  (b) $\alpha = R_{\beta 1}^{-1}$, $\beta = P_1 R_{\beta 1}^{-1}$ *for* $\sigma = (13), (123)$,
  (c) $\alpha = P_1^{-1} L_{\alpha 1}^{-1}$, $\beta = L_{\alpha 1}^{-1}$ *for* $\sigma = (23), (132)$,

  where $\alpha 1 \cdot \beta 1 = 1$.

Proof. Case 1. Putting $f = e = 1$ in the case (1) of Lemma 3.1 we obtain the following:

If $\sigma = \varepsilon$ we get $\alpha x \cdot 1 = \gamma x$, i.e. $\alpha x = \gamma x$ and $\alpha 1 \cdot x = \gamma x$, i.e. $\gamma x = L_{\alpha 1} x$ $\forall\, x \in Q$. Hence, $\alpha = \gamma = L_{\alpha 1}$.

If $\sigma = (12)$ we get $\alpha 1 \cdot x = \gamma x$, i.e. $\gamma x = L_{\alpha 1} x$ and $\alpha x \cdot 1 = \gamma x$, i.e. $\alpha x = \gamma x$ $\forall\, x \in Q$. Hence, $\alpha = \gamma = L_{\alpha 1}$.

If $\sigma = (13)$ we get $\alpha x \cdot 1 = \gamma x$, i.e. $\alpha x = \gamma x$ and $\alpha x \cdot x = \gamma 1$, i.e. $\alpha x = P_{\gamma 1}^{-1}$ $\forall\, x \in Q$. Also we have $\alpha 1 = \gamma 1$. Hence, $\alpha = \gamma = P_{\alpha 1}^{-1}$.

If $\sigma = (23)$ we get $\alpha x \cdot x = \gamma 1$, i.e. $\alpha x = P_{\gamma 1}^{-1}$ and $\alpha 1 \cdot x = \gamma x$, i.e. $\gamma x = L_{\alpha 1} x$ $\forall\, x \in Q$. Also we have $\alpha 1 = \gamma 1$. Hence, $\alpha = P_{\alpha 1}^{-1}$ and $\gamma = L_{\alpha 1}$.

If $\sigma = (123)$ we get $\alpha x \cdot x = \gamma 1$, i.e. $\alpha x = P_{\gamma 1}^{-1}$ and $\alpha x \cdot 1 = \gamma x$, i.e. $\alpha x = \gamma x$ $\forall\, x \in Q$. Also we have $\alpha 1 = \gamma 1$. Hence, $\alpha = \gamma = P_{\alpha 1}^{-1}$.

If $\sigma = (132)$ we get $\alpha 1 \cdot x = \gamma x$, i.e. $\gamma x = L_{\alpha 1} x$ and $\alpha x \cdot x = \gamma 1$, i.e. $\alpha x = P_{\gamma 1}^{-1}$ $\forall\, x \in Q$. Also we have $\alpha 1 = \gamma 1$. Hence, $\gamma = L_{\alpha 1}$ and $\alpha = P_{\alpha 1}^{-1}$.

Other cases can be proved similarly. □

Remark 3.2. Since for $\sigma = \varepsilon$, ε-A-nuclei coincide with A-nuclei. Therefore, in Theorem 3.4 (3)(a) $\alpha = R_{\beta 1}$, $\beta = L_{\beta 1}^{-1}$ [11].

Remark 3.3. From Theorem 3.4 and Remark 3.2 we can say that in a loop $(Q, \cdot)$ with identity element 1 the following hold:

(1) any element of the left ε, (12), (13), (123), (23) and (132)-A-nucleus is of the form $(\varepsilon, (L_a, \varepsilon, L_a))$, $((12), (L_a, \varepsilon, L_a))$, $((13), (P_a^{-1}, \varepsilon, P_a^{-1}))$, $((123), (P_a^{-1}, \varepsilon, P_a^{-1}))$, $((23), (P_a^{-1}, \varepsilon, L_a))$ and $((132), (P_a^{-1}, \varepsilon, L_a))$, respectively, where a is a fixed element of Q.

(2) any element of the right ε, (12), (13), (123), (23) and (132)-A-nucleus is of the form $(\varepsilon,(\varepsilon,R_a,R_a))$, $((12),(\varepsilon,R_a,R_a))$, $((13),(\varepsilon,P_a,R_a))$, $((123),(\varepsilon,P_a,R_a))$, $((23),(\varepsilon,P_a,P_a))$ and $((132),(\varepsilon,P_a,P_a))$, respectively, where a is a fixed element of Q.

(3) any element of the middle ε, (12), (13), (123), (23) and (132)-A-nucleus is of the form $(\varepsilon,(R_b,L_b^{-1},\varepsilon))$, $((12),(R_b^{-1},L_a^{-1},\varepsilon))$ with $a\cdot b = 1$, $((13),(R_b^{-1},P_1^{-1}R_b^{-1},\varepsilon))$, $((123),(R_b^{-1},P_1^{-1}R_b^{-1},\varepsilon))$, $((23),(P_1^{-1}L_a^{-1},L_a^{-1},\varepsilon)$ and $((132),(P_1^{-1}L_a^{-1},L_a^{-1},\varepsilon))$, respectively, where a and b are fixed elements of Q.

Corollary 3.5. *In any loop $(Q,\cdot)$ we have $|{}^{\sigma}N_l^A| = |{}^{\sigma}N_r^A| = |{}^{\sigma}N_m^A| = |Q|$ for all $\sigma\in S_3$.*

Proof. In any quasigroup $(Q,\cdot)$, if T_a is a left/ right/ middle translation then $a\neq b \Rightarrow T_a\neq T_b$ for all $a,b\in Q$. Hence from Theorem 3.4 in a loop $(Q,\cdot)$ for different elements in Q, we get different elements in left/ right/ middle σ-A-nucleus. □

Theorem 3.5. *Let $(Q,\cdot)$ be a unipotent left loop and $\sigma\in S_3$.*

(1) *If $(\sigma,(\alpha,\varepsilon,\gamma))\in \mathrm{Aus}(Q,\cdot)$ then*:
   (a) $\alpha = P_{\alpha 1\cdot 1}^{-1}$, $\gamma = L_{\alpha 1}$ *for* $\sigma=\varepsilon,(13)$,
   (b) $\alpha = P_{\gamma 1}^{-1}$, $\gamma = R_1P_{\gamma 1}^{-1}$ *for* $\sigma = (12),(132)$,
   (c) $\alpha = R_1^{-1}L_{\alpha 1}$, $\gamma = L_{\alpha 1}$ *for* $\sigma=(23),(123)$.

(2) *If $(\sigma,(\varepsilon,\beta,\gamma))\in \mathrm{Aus}(Q,\cdot)$ then*:
   (a) $\beta=\gamma=P_{\beta 1}$ *for* $\sigma=\varepsilon,(13)$,
   (b) $\beta = P_{\beta 1}$, $\gamma = R_{\beta 1}$ *for* $\sigma=(12),(132)$,
   (c) $\beta=\gamma=R_{\beta 1}$ *for* $\sigma=(23),(123)$.

(3) *If $(\sigma,(\alpha,\beta,\varepsilon))\in \mathrm{Aus}(Q,\cdot)$ then*:
   (a) $\alpha=\beta=L_{\alpha 1}^{-1}$ *for* $\sigma=\varepsilon,(13)$,
   (b) $\alpha=\beta=R_{\alpha 1}^{-1}$ *for* $\sigma=(12),(132)$,
   (c) $\alpha = R_{\alpha 1}^{-1}$, $\beta = L_{\alpha 1}^{-1}$ *for* $\sigma=(23),(123)$,

Proof. This can be proved on similar lines as in Theorem 3.4 by putting $f=s=1$ in Lemma 3.1 (or follows from Corollaries 3.2 and 3.4). □

Remark 3.4. Since for $\sigma=\varepsilon$, ε-A-nuclei coincides with A-nuclei. Therefore, in Theorem 3.5 (1)(a) $\alpha = P_{\alpha 1}$, $\gamma = L_{\alpha 1}$ [11].

Remark 3.5. From Theorem 3.5 and Remark 3.4 we can say that in a unipotent left loop $(Q,\cdot)$ with left identity element $f=1$ the following hold:

(1) any element of the left ε, (13), (12), (132), (23) and (123)-A-nucleus is of the form $(\varepsilon,(P_a,\varepsilon,L_a))$, $((13),(P_{a\cdot 1}^{-1},\varepsilon,L_a))$, $((12),(P_b^{-1},\varepsilon,R_1P_b^{-1}))$, $((132),(P_b^{-1},\varepsilon,R_1P_b^{-1}))$, $((23),(R_1^{-1}L_a,\varepsilon,L_a))$ and $((123),(R_1^{-1}L_a,\varepsilon,L_a))$, respectively, where a and b are fixed elements of Q.

(2) any element of the right ε, (13), (12), (132), (23) and (123)-A-nucleus is of the form $(\varepsilon,(\varepsilon,P_a,P_a))$, $((13),(\varepsilon,P_a,P_a))$, $((12),(\varepsilon,P_a,R_a))$, $((132),(\varepsilon,P_a,R_a))$, $((23),(\varepsilon,R_a,R_a))$ and $((123),(\varepsilon,R_a,R_a))$, respectively, where a is a fixed element of Q.

(3) any element of the middle ε, (13), (12), (132), (23) and (123)-A-nucleus is of the form $(\varepsilon,(L_a^{-1},L_a^{-1},\varepsilon))$, $((13),(L_a^{-1},L_a^{-1},\varepsilon))$, $((12),(R_a^{-1},R_a^{-1},\varepsilon))$, $((132),(R_a^{-1},R_a^{-1},\varepsilon))$, $((23),(R_a^{-1},L_a^{-1},\varepsilon)$ and $((123),(R_a^{-1},L_a^{-1},\varepsilon))$, respectively, where a is a fixed element of Q.

Corollary 3.6. *In any unipotent left loop $(Q,\cdot)$ we have $|{}^{\sigma}N_l^A| = |{}^{\sigma}N_r^A| = |{}^{\sigma}N_m^A| = |Q|$ for all $\sigma \in S_3$.*

Proof. This can be proved on similar lines as in Corollary 3.5 using Theorem 3.5. □

Theorem 3.6. *Let $(Q,\cdot)$ be a unipotent right loop and $\sigma \in S_3$.*

(1) *If* $(\sigma,(\alpha,\varepsilon,\gamma)) \in \mathrm{Aus}(Q,\cdot)$ *then*:
   (a) $\alpha = \gamma = P_{\alpha 1}^{-1}$ *for* $\sigma = \varepsilon,(23)$,
   (b) $\alpha = L_{\alpha 1}$, $\gamma = P_{\alpha 1}^{-1}$ *for* $\sigma = (12),(123)$,
   (c) $\alpha = \gamma = L_{\alpha 1}$ *for* $\sigma = (13),(132)$.

(2) *If* $(\sigma,(\varepsilon,\beta,\gamma)) \in \mathrm{Aus}(Q,\cdot)$ *then*:
   (a) $\beta = P_{1\cdot\beta 1}$, $\gamma = R_{\beta 1}$ *for* $\sigma = \varepsilon,(23)$,
   (b) $\beta = P_{\gamma 1}$, $\gamma = L_1 P_{\gamma 1}$ *for* $\sigma = (12),(123)$,
   (c) $\beta = L_1^{-1}R_{\beta 1}$, $\gamma = R_{\beta 1}$ *for* $\sigma = (13),(132)$.

(3) *If* $(\sigma,(\alpha,\beta,\varepsilon)) \in \mathrm{Aus}(Q,\cdot)$ *then*:
   (a) $\alpha = \beta = R_{\alpha 1}^{-1}$ *for* $\sigma = \varepsilon,(23)$,
   (b) $\alpha = \beta = L_{\alpha 1}^{-1}$ *for* $\sigma = (12),(123)$,
   (c) $\alpha = R_{\alpha 1}^{-1}$, $\beta = L_{\alpha 1}^{-1}$ *for* $\sigma = (13),(132)$.

Proof. This can be proved on similar lines as in Theorem 3.4 by putting $e = s = 1$ in Lemma 3.1 (or follows from Corollaries 3.3 and 3.4). □

Remark 3.6. Since for $\sigma = \varepsilon$, ε-A-nuclei coincides with A-nuclei. Therefore, in Theorem 3.6 (2)(a) $\beta = P_{\beta 1}^{-1}$, $\gamma = R_{\beta 1}$ [11].

Remark 3.7. From Theorem 3.6 and Remark 3.6 we can say that in a unipotent right loop $(Q,\cdot)$ with right identity element $e = 1$ the following hold:

(1) any element of the left ε, (23), (12), (123), (13) and (132)-A-nucleus is of the form $(\varepsilon,(P_a^{-1},\varepsilon,P_a^{-1}))$, $((23),(P_a^{-1},\varepsilon,P_a^{-1}))$, $((12),(L_a,\varepsilon,P_a^{-1}))$, $((123),(L_a,\varepsilon,P_a^{-1}))$, $((13),(L_a,\varepsilon,L_a))$ and $((132),(L_a,\varepsilon,L_a))$, respectively, where a is a fixed element of Q.

(2) any element of the right ε, (23), (12), (123), (13) and (132)-A-nucleus is of the form $(\varepsilon,(\varepsilon,P_a^{-1},R_a))$, $((23),(\varepsilon,P_{1\cdot a},R_a))$, $((12),(\varepsilon,P_b,L_1P_b))$, $((123),(\varepsilon,P_b,L_1P_b))$, $((13),(\varepsilon,L_1^{-1}R_a,R_a))$ and $((132),(\varepsilon,L_1^{-1}R_a,R_a))$, respectively, where a and b are fixed elements of Q.

(3) any element of the middle ε, (23), (12), (123), (13) and (132)-A-nucleus is of the form $(\varepsilon,(R_a^{-1},R_a^{-1},\varepsilon))$, $((23),(R_a^{-1},R_a^{-1},\varepsilon))$, $((12),(L_a^{-1},L_a^{-1},\varepsilon))$, $((123),(L_a^{-1},L_a^{-1},\varepsilon))$, $((13),(R_a^{-1},L_a^{-1},\varepsilon)$ and $((132),(R_a^{-1},L_a^{-1},\varepsilon))$, respectively, where a is a fixed element of Q.

Corollary 3.7. *In any unipotent right loop $(Q,\cdot)$ we have $|{}^{\sigma}N_l^A| = |{}^{\sigma}N_r^A| = |{}^{\sigma}N_m^A| = |Q|$ for all $\sigma\in S_3$.*

Proof. This can be proved on similar lines as in Corollary 3.5 using Theorem 3.6. □

In the following theorem, we derive left, right and middle ε-A-nuclei (or A-nuclei) of a quasigroup which is an isostrophic image of a loop in terms of translation maps.

Theorem 3.7. *Let $(Q,*)$ is a loop and $(Q,\cdot)=(Q,*)(\sigma,(\alpha,\beta,\gamma))$.*

(1) *If $\sigma=\varepsilon,(12)$ and $\gamma=\varepsilon$ then:*
   (a) $N_l^A(Q,\cdot)=(R_a^{-1}L_{R_a^{-1}l}L_b^{-1}R_a,\varepsilon,L_{R_a^{-1}l}L_b^{-1})$,
   (b) $N_r^A(Q,\cdot)=(\varepsilon,L_b^{-1}R_{L_b^{-1}r}R_a^{-1}L_b,R_{L_b^{-1}r}R_a^{-1})$,
   (c) $N_m^A(Q,\cdot)=(R_a^{-1}R_{L_b^{-1}m},L_{R_a^{-1}m}^{-1}L_b,\varepsilon)$,

   where a,b are fixed elements of Q, $l\in N_l(Q,*^{\sigma})$, $r\in N_r(Q,*^{\sigma})$ and $m\in N_m(Q,*^{\sigma})$. Here $(Q,*^{\sigma})$ is the σ parastrophe of $(Q,*)$.

(2) *If $\sigma=(13),(132)$ and $\alpha=\varepsilon$ then:*
   (a) $N_l^A(Q,\cdot)=(P_{R_bc}^{-1}P_a,\varepsilon,R_bP_{R_bc}^{-1}P_aR_b^{-1})$,
   (b) $N_r^A(Q,\cdot)=(\varepsilon,P_aP_{R_bc}^{-1},R_{P_ac}R_b^{-1})$,
   (c) $N_m^A(Q,\cdot)=(R_{P_ac}^{-1}R_b,P_aR_{P_ac}^{-1}R_bP_a^{-1},\varepsilon)$,

   where a,b,c are fixed elements of Q.

(3) *If $\sigma=(23),(123)$ and $\beta=\varepsilon$ then:*
   (a) $N_l^A(Q,\cdot)=(P_a^{-1}P_{L_b^{-1}c},\varepsilon,L_{P_a^{-1}c}L_b^{-1})$,
   (b) $N_r^A(Q,\cdot)=(\varepsilon,P_{L_b^{-1}c}P_a^{-1},L_bP_{L_b^{-1}c}P_a^{-1}L_b^{-1})$,
   (c) $N_m^A(Q,\cdot)=(P_a^{-1}L_{P_a^{-1}c}^{-1}L_bP_a,L_{P_a^{-1}c}^{-1}L_b,\varepsilon)$,

   where a,b,c are fixed elements of Q.

Proof. Case 1 (a): If $\sigma=(12)$ and $\gamma=\varepsilon$, take a quasigroup $(Q,\circ)$ such that $(Q,\circ)=(Q,*)(12)(\varepsilon,\varepsilon,\varepsilon)$. From Table 1, the (12) parastrophe of a loop is again a loop. Therefore, the quasigroup $(Q,\circ)$ is a loop. We get $(Q,\cdot)=(Q,\circ)(\alpha,\beta,\varepsilon)$ and $(Q,\circ)$ is a loop. This can be written as $(Q,\circ)=(Q,\cdot)(\alpha^{-1},\beta^{-1},\varepsilon)$ which implies $x\circ y=\alpha^{-1}x\cdot\beta^{-1}y$. If $x=1$, then we have $1\circ y=y=\alpha^{-1}1\cdot\beta^{-1}y$. Therefore,

$y = L_{\alpha^{-1}1}\beta^{-1}y$, $\beta^{-1} = L^{-1}_{\alpha^{-1}1} = L^{-1}_b$ (say), where $b = \alpha^{-1}1$. If we take $y = 1$, then $x \circ 1 = x = \alpha^{-1}x \cdot \beta^{-1}1$. Therefore, $x = R_{\beta^{-1}1}\alpha^{-1}x$, i.e. $\alpha^{-1} = R^{-1}_{\beta^{-1}1} = R^{-1}_a$ (say), where $a = \beta^{-1}1$. We get $(Q, \circ) = (Q, \cdot)T$; $T = (R^{-1}_a, L^{-1}_b, \varepsilon)$. This gives $x \circ y = R^{-1}_a x \cdot L^{-1}_b y = L_{R^{-1}_a x}L^{-1}_b y$. Therefore, we have

$$L^{\circ}_x = L_{R^{-1}_a x}L^{-1}_b. \tag{3.2}$$

Similarly

$$R^{\circ}_y = R_{L^{-1}_b y}R^{-1}_a. \tag{3.3}$$

From Theorem 2.2(1), any element of the left A-nucleus of the loop $(Q, \circ)$ has the form $(L^{\circ}_l, \varepsilon, L^{\circ}_l)$, where $l \in N_l(Q, \circ) = N_l(Q, *^{\sigma})$. Since $(Q, \circ) = (Q, *)(12)(\varepsilon, \varepsilon, \varepsilon)$, therefore from Table 2, we get ${}_2N^*_r = {}_1N^{\circ}_l = L^{\circ}_l$, ${}_3N^*_r = {}_3N^{\circ}_l = L^{\circ}_l$. Hence from Eq. (3.2), $N^A_r(Q, *) = (\varepsilon, L_{R^{-1}_a l}L^{-1}_b, L_{R^{-1}_a l}L^{-1}_b)$. We have $(Q, \cdot) = (Q, *)((12), (R_a, L_b, \varepsilon))$, thus from Theorem 2.1 we get $N^A_l(Q, \cdot) = S^{-1}N^A_r(Q, *)S$, where $S = ((12), (R_a, L_b, \varepsilon))$. Therefore, $N^A_l(Q, \cdot) = ((12), (L^{-1}_b, R^{-1}_a, \varepsilon))(\varepsilon, L_{R^{-1}_a l}L^{-1}_b, L_{R^{-1}_a l}L^{-1}_b)((12), (R_a, L_b, \varepsilon))) = (\varepsilon, (R^{-1}_a L_{R^{-1}_a l}L^{-1}_b R_a, \varepsilon, L_{R^{-1}_a l}L^{-1}_b))$. Hence, $N^A_l(Q, \cdot) = (R^{-1}_a L_{R^{-1}_a l}L^{-1}_b R_a, \varepsilon, L_{R^{-1}_a l}$ $L^{-1}_b)$.

If $\sigma = \varepsilon$ and $\gamma = \varepsilon$, similarly we get $(Q, \cdot) = (Q, *)(R_a, L_b, \varepsilon)$. Again from Theorem 2.2(1) and Eq. (3.2), any element of the left A-nucleus of the loop $(Q, *)$ has the form $(L_{R^{-1}_a l}L^{-1}_b, \varepsilon, L_{R^{-1}_a l}L^{-1}_b)$, where $l \in N_l(Q, *)$. Since from Theorem 2.1 we have $N^A_l(Q, \cdot) = T^{-1}N^A_l(Q, *)T$, where $T = (R_a, L_b, \varepsilon)$. Therefore, $N^A_l(Q, \cdot) = (R^{-1}_a, L^{-1}_b, \varepsilon)(L_{R^{-1}_a l}L^{-1}_b, \varepsilon, L_{R^{-1}_a l}L^{-1}_b)(R_a, L_b, \varepsilon) = (R^{-1}_a L_{R^{-1}_a l}L^{-1}_b R_a, \varepsilon, L_{R^{-1}_a l}L^{-1}_b)$. Hence, $N^A_l(Q, \cdot) = (R^{-1}_a L_{R^{-1}_a l}L^{-1}_b R_a, \varepsilon, L_{R^{-1}_a l}$ $L^{-1}_b)$.

Cases (b) and (c) can be proved similarly.

Case 2 (b): If $\sigma = (13)$ and $\alpha = \varepsilon$, take a quasigroup $(Q, \circ)$ such that $(Q, \circ) = (Q, *)(13)(\varepsilon, \varepsilon, \varepsilon)$. From Table 1, the (13) parastrophe of a loop is a unipotent right loop. Therefore, the quasigroup $(Q, \circ)$ is a unipotent right loop. We get $(Q, \cdot) = (Q, \circ)(\varepsilon, \beta, \gamma)$ and $(Q, \circ)$ is a unipotent right loop. This can be written as $(Q, \circ) = (Q, \cdot)(\varepsilon, \beta^{-1}, \gamma^{-1})$ which implies $x \circ y = \gamma(x \cdot \beta^{-1}y)$. If $y = 1$, then we have $x \circ 1 = x = \gamma(x \cdot \beta^{-1}1)$. Therefore, $\gamma^{-1}x = x \cdot \beta^{-1}1$, $\gamma^{-1} = R_{\beta^{-1}1} = R_b$ (say), where $b = \beta^{-1}1$. If we take $x = y$, then $x \circ x = 1 = \gamma(x \cdot \beta^{-1}x)$. Therefore, $\gamma^{-1}1 = x \cdot \beta^{-1}x$, $\beta^{-1}x = x\backslash\gamma^{-1}1 = P_{\gamma^{-1}1}x$, i.e. $\beta^{-1} = P_{\gamma^{-1}1} = P_a$ (say), where $a = \gamma^{-1}1$. We get $(Q, \circ) = (Q, \cdot)T$; $T = (\varepsilon, P_a, R_b)$. This gives $x \circ y = R^{-1}_b(x \cdot P_a y) = R^{-1}_b(L_x P_a y)$. Therefore, we have

$$L^{\circ}_x = R^{-1}_b L_x P_a. \tag{3.4}$$

Similarly

$$R^{\circ}_y = R^{-1}_b R_{P_a y}. \tag{3.5}$$

Let $x \circ y = R_b^{-1}(x \cdot P_a y) = z$. Then $x \circ y = z, R_b^{-1}(x \cdot P_a y) = z$, i.e. $z/y = x, R_b z/P_a y = x$. Therefore, $P_z^{\circ -1} y = P_{R_b z}^{-1} P_a y$, i.e.

$$P_z^{\circ -1} = P_{R_b z}^{-1} P_a. \tag{3.6}$$

From Remark 3.7, any element of the right A-nucleus of the unipotent right loop $(Q, \circ)$ has the form $(\varepsilon, P_c^{\circ -1}, R_c^\circ)$, where c is a fixed element of Q. Since $(Q, \circ) = (Q, *)(13)(\varepsilon, \varepsilon, \varepsilon)$, therefore from Table 2, we get ${}_1N_m^* = {}_3N_r^\circ = R_c^\circ$, ${}_2N_m^* = {}_2N_r^\circ = P_c^{\circ -1}$. Hence from Eqs. (3.5) and (3.6), any middle A-nucleus of $(Q, *)$ has element of the form $(R_b^{-1} R_{P_a c}, P_{R_b c}^{-1} P_a, \varepsilon)$. We have $(Q, \cdot) = (Q, *)((13), (\varepsilon, P_a^{-1}, R_b^{-1}))$, thus from Theorem 2.1 we get $N_r^A(Q, \cdot) = S^{-1} N_m^A(Q, *) S$, where $S = ((13), (\varepsilon, P_a^{-1}, R_b^{-1}))$. Therefore, $N_r^A(Q, \cdot) = ((13), (R_b, P_a, \varepsilon))(\varepsilon, (R_b^{-1} R_{P_a c}, P_{R_b c}^{-1} P_a, \varepsilon))((13), (\varepsilon, P_a^{-1}, R_b^{-1}))) = (\varepsilon, (\varepsilon, P_a P_{R_b c}^{-1}, R_{P_a c} R_b^{-1}))$. Hence, $N_r^A(Q, \cdot) = (\varepsilon, P_a P_{R_b c}^{-1}, R_{P_a c} R_b^{-1})$.

If $\sigma = (132)$ and $\alpha = \varepsilon$, take a quasigroup $(Q, \circ)$ such that $(Q, \circ) = (Q, *)(132)(\varepsilon, \varepsilon, \varepsilon)$. From Table 1, the (132) parastrophe of a loop is also a unipotent right loop. Therefore, the quasigroup $(Q, \circ)$ is a unipotent right loop. We get $(Q, \cdot) = (Q, \circ)(\varepsilon, \beta, \gamma)$ and $(Q, \circ)$ is a unipotent right loop. Following the same procedure as above we get $(Q, \cdot) = (Q, *)((132), (\varepsilon, P_a^{-1}, R_b^{-1}))$. Therefore from Theorem 2.1, we get $N_r^A(Q, \cdot) = S^{-1} N_l^A(Q, *) S$, where $S = ((132), (\varepsilon, P_a^{-1}, R_b^{-1}))$. Since $(Q, \circ) = (Q, *)(132)(\varepsilon, \varepsilon, \varepsilon)$, therefore from Table 2, we have components of left A-nucleus of $(Q, *)$ from the components of right A-nucleus of $(Q, \circ)$ as: ${}_1N_l^* = {}_3N_r^\circ = R_c^\circ$, ${}_3N_l^* = {}_2N_r^\circ = P_c^{\circ -1}$. Hence from Eqs. (3.5) and (3.6), $N_m^A(Q, *) = (R_b^{-1} R_{P_a c}, \varepsilon, P_{R_b c}^{-1} P_a)$. Therefore, $N_r^A(Q, \cdot) = ((123), (R_b, \varepsilon, P_a))(\varepsilon, (R_b^{-1} R_{P_a c}, \varepsilon, P_{R_b c}^{-1} P_a))((132), (\varepsilon, P_a^{-1}, R_b^{-1}))) = (\varepsilon, (\varepsilon, P_a P_{R_b c}^{-1}, R_{P_a c} R_b^{-1}))$. Hence, $N_r^A(Q, \cdot) = (\varepsilon, P_a P_{R_b c}^{-1}, R_{P_a c} R_b^{-1})$.

Cases (a) and (c) can be proved similarly.

Case 3 (c): If $\sigma = (23)$ and $\beta = \varepsilon$, take a quasigroup $(Q, \circ)$ such that $(Q, \circ) = (Q, *)(23)(\varepsilon, \varepsilon, \varepsilon)$. From Table 1, the (23) parastrophe of a loop is a unipotent left loop. Therefore, the quasigroup $(Q, \circ)$ is a unipotent left loop. We get $(Q, \cdot) = (Q, \circ)(\alpha, \varepsilon, \gamma)$ and $(Q, \circ)$ is a unipotent left loop. This can be written as $(Q, \circ) = (Q, \cdot)(\alpha^{-1}, \varepsilon, \gamma^{-1})$ which implies $x \circ y = \gamma(\alpha^{-1} x \cdot y)$. If $x = 1$, then we have $1 \circ y = y = \gamma(\alpha^{-1} 1 \cdot y)$. Therefore, $\gamma^{-1} y = \alpha^{-1} 1 \cdot y$, $\gamma^{-1} = L_{\alpha^{-1} 1} = L_b$ (say), where $b = \alpha^{-1} 1$. If we take $x = y$, then $x \circ x = 1 = \gamma(\alpha^{-1} x \cdot x)$. Therefore, $\gamma^{-1} 1 = \alpha^{-1} x \cdot x$, $\alpha^{-1} x = \gamma^{-1} 1 / x = P_{\gamma^{-1} 1}^{-1} x$, i.e. $\alpha^{-1} = P_{\gamma^{-1} 1}^{-1} = P_a^{-1}$ (say), where $a = \gamma^{-1} 1$. We get $(Q, \circ) = (Q, \cdot) T\,; T = (P_a^{-1}, \varepsilon, L_b)$. This gives $x \circ y = L_b^{-1}(P_a^{-1} x \cdot y) = L_b^{-1}(L_{P_a^{-1} x} y)$. Therefore, we have

$$L_x^\circ = L_b^{-1} L_{P_a^{-1} x}. \tag{3.7}$$

Similarly

$$R_y^\circ = L_b^{-1} R_y P_a^{-1}. \tag{3.8}$$

Let $z = x \circ y = L_b^{-1}(P_a^{-1}x \cdot y)$. Then $x \circ y = z, L_b^{-1}(P_a^{-1}x \cdot y) = z$, i.e. $x\backslash z = y, P_a^{-1}x\backslash L_b^{-1}z = y$. Therefore, $P_z^\circ x = P_{L_b^{-1}z}P_a^{-1}x$, i.e.

$$P_z^\circ = P_{L_b^{-1}z}P_a^{-1}. \tag{3.9}$$

From Remark 3.5, any element of the middle A-nucleus of the unipotent left loop $(Q, \circ)$ has the form $(L_c^{\circ-1}, L_c^{\circ-1}, \varepsilon)$, where c is a fixed element of Q. Since $(Q, \circ) = (Q, *)(23)(\varepsilon, \varepsilon, \varepsilon)$, therefore from Table 2, we get ${}_1N_l^* = {}_1N_m^\circ = L_c^{\circ-1}$, ${}_3N_l^* = {}_2N_m^\circ = L_c^{\circ-1}$. Hence from Eq. (3.7), $N_l^A(Q, *) = (L_{P_a^{-1}c}^{-1}L_b, \varepsilon, L_{P_a^{-1}c}^{-1}L_b)$. We have $(Q, \cdot) = (Q, *)((23), (P_a, \varepsilon, L_b^{-1}))$, thus from Theorem 2.1 we get $N_m^A(Q, \cdot) = S^{-1}N_l^A(Q, *)S$, where $S = ((23), (P_a, \varepsilon, L_b^{-1}))$. Therefore, $N_m^A(Q, \cdot) = ((23), (P_a^{-1}, L_b, \varepsilon))(\varepsilon, (L_{P_a^{-1}c}^{-1}L_b, \varepsilon, L_{P_a^{-1}c}^{-1}L_b))((23), (P_a, \varepsilon, L_b^{-1})) = (\varepsilon, (P_a^{-1}L_{P_a^{-1}c}^{-1}L_bP_a, L_{P_a^{-1}c}^{-1}L_b, \varepsilon))$. Hence, $N_m^A(Q, \cdot) = (P_a^{-1}L_{P_a^{-1}c}^{-1}L_bP_a, L_{P_a^{-1}c}^{-1}L_b, \varepsilon)$.

If $\sigma = (123)$ and $\alpha = \varepsilon$, take a quasigroup $(Q, \circ)$ such that $(Q, \circ) = (Q, *)(123)(\varepsilon, \varepsilon, \varepsilon)$. From Table 1, the (123) parastrophe of a loop is a unipotent left loop. Therefore, the quasigroup $(Q, \circ)$ is a unipotent left loop. We get $(Q, \cdot) = (Q, \circ)(\varepsilon, \beta, \gamma)$ and $(Q, \circ)$ is a unipotent left loop. Following the same procedure as above we get $(Q, \cdot) = (Q, *)((123), (P_a, \varepsilon, L_b^{-1}))$. Therefore from Theorem 2.1, we get $N_m^A(Q, \cdot) = S^{-1}N_l^A(Q, *)S$, where $S = ((123), (P_a, \varepsilon, L_b^{-1}))$. Since $(Q, \circ) = (Q, *)(123)(\varepsilon, \varepsilon, \varepsilon)$, therefore from Table 2, we have components of left A-nucleus of $(Q, *)$ from the components of middle A-nucleus of $(Q, \circ)$ as: ${}_1N_l^* = {}_2N_m^\circ = L_c^{\circ-1}$, ${}_3N_l^* = {}_1N_m^\circ = L_c^{\circ-1}$. Hence from Eq. (3.7), $N_l^A(Q, *) = (L_{P_a^{-1}c}^{-1}L_b, \varepsilon, L_{P_a^{-1}c}^{-1}L_b)$. Therefore, $N_m^A(Q, \cdot) = ((132), (\varepsilon, L_b, P_a^{-1}))(\varepsilon, (L_{P_a^{-1}c}^{-1}L_b, \varepsilon, L_{P_a^{-1}c}^{-1}L_b))((123), (P_a, \varepsilon, L_b^{-1}))) = (\varepsilon, (P_a^{-1}L_{P_a^{-1}c}^{-1}L_bP_a, L_{P_a^{-1}c}^{-1}L_b, \varepsilon))$. Hence, $N_m^A(Q, \cdot) = (P_a^{-1}L_{P_a^{-1}c}^{-1}L_bP_a, L_{P_a^{-1}c}^{-1}L_b, \varepsilon)$.

Cases (a) and (b) can be proved similarly. □

Theorem 3.8. *If $(Q, \cdot)$ and $(Q, *)$ are quasigroups such that $(Q, *) = (Q, \cdot)(\varepsilon, P_a, R_b)$ then the following hold*:

(1) $N_l^A(Q, \cdot) = (P_{R_bc}^{-1}P_a, \varepsilon, R_bP_{R_bc}^{-1}P_aR_b^{-1})$,
(2) $N_r^A(Q, \cdot) = (\varepsilon, P_aP_{R_bc}^{-1}, R_{P_ac}R_b^{-1})$,
(3) $N_m^A(Q, \cdot) = (R_{P_ac}^{-1}R_b, P_aR_{P_ac}^{-1}R_bP_a^{-1}, \varepsilon)$,

where $a, b, c \in Q$.

Proof. Given that $(Q, *) = (Q, \cdot)(\varepsilon, P_a, R_b)$. Therefore, the quasigroup $(Q, *)$ is a unipotent right loop [3]. The result can be proved on the similar line as that of Theorem 3.7 (2) and hence omitted. □

Theorem 3.9. *If $(Q, \cdot)$ and $(Q, *)$ are quasigroups such that $(Q, *) = (Q, \cdot)(P_a^{-1}, \varepsilon, L_b)$ then the following hold*:

(1) $N_l^A(Q, \cdot) = (P_a^{-1}P_{L_b^{-1}c}, \varepsilon, L_{P_a^{-1}c}L_b^{-1})$,

(2) $N_r^A(Q,\cdot) = (\varepsilon, P_{L_b^{-1}c}P_a^{-1}, L_bP_{L_b^{-1}c}P_a^{-1}L_b^{-1})$,
(3) $N_m^A(Q,\cdot) = (P_a^{-1}L_{P_a^{-1}c}^{-1}L_bP_a, L_{P_a^{-1}c}^{-1}L_b, \varepsilon)$,

where $a, b, c \in Q$.

Proof. Given that $(Q,*) = (Q,\cdot)(P_a^{-1}, \varepsilon, L_b)$. Therefore, the quasigroup $(Q,*)$ is a unipotent left loop [3]. The result can be proved on the similar line as that of Theorem 3.7 (3) and hence omitted. □

4. Conclusions

In this paper, we have discussed some concepts required for the understanding of the paper and redefined σ-A-nuclei of a quasigroup as a generalization of A-nuclei. Connections between the components of σ-A-nuclei and local identity elements of a quasigroup have been derived. Further, we have found the explicit structure of σ-A-nuclei of a loop, a unipotent left loop and a unipotent right loop in terms of translation maps. Finally, the left, right and middle A-nuclei of a quasigroup which is an isostrophic image of a loop have been found.

This study may further be carried out to obtain σ-A-nuclei of various quasigroup classes available in the literature.

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