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A class of two-level implicit unconditionally stable methods for a fourth order parabolic equation☆

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ABSTRACT

In this paper, we discuss two-level implicit methods for the solution of a special type of fourth order parabolic partial differential equation of the form u_{xxxx}-$2u_{xxt}$+u_{tt}=$f(x,t,u)$, $0 < x < 1$, $t > 0$ subject to appropriate initial and Dirichlet boundary conditions by converting the original problem to a coupled system of two second order parabolic equations. We use only three spatial grid points and it is not required to discretize the boundary conditions. The proposed Crank-Nicolson type scheme is second order accurate in both the temporal and spatial dimensions while the compact Crandall's type scheme is second order accurate in temporal and fourth order accurate in spatial dimension. The methods do not require any fictitious nodes outside the solution domain for handling the boundary conditions. For a fixed mesh ratio parameter ($\Delta t/\Delta x^2$), the proposed Crandall's type method behaves like a fourth order method in space. Using matrix stability analysis, the proposed methods are shown to be unconditionally stable. The resulting implicit difference formulas gives block tri-diagonal matrix structure which is solved efficiently using block Gauss-Seidel method or block Newton method depending on linear or nonlinear behaviour of the equations. The methods compute the numerical value of u and time-dependent Laplacian $u_{xx}-u_t$, simultaneously. Numerical results are provided to demonstrate the accuracy and efficiency of the proposed methods.

1. Introduction

We consider a special type fourth order nonlinear parabolic equation of the form

$$\frac{\partial^4 u}{\partial x^4} - 2\frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial^2 u}{\partial t^2} = f(x,t,u),\ 0 < x < 1, t > 0 \tag{1}$$

The initial and boundary conditions are given by

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x),\ 0 \le x \le 1 \tag{2}$$

$$u(0,t) = b_0(t), \quad u_{xx}(0,t) = c_0(t), \quad u(1,t) = b_1(t), \quad u_{xx}(1,t) = c_1(t), \quad t > 0 \tag{3}$$

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We assume that the functions $f(x, t, u)$, $\phi(x)$, $\psi(x)$, $b_0(t)$, $b_1(t)$, $c_0(t)$ and $c_1(t)$ are sufficiently smooth and their required higher order derivatives exist.

Fourth order parabolic partial differential equations (PDEs) are often encountered in various physical phenomena such as in modelling of the clamped rod problem, in the study of viscoelastic and inelastic fluid flow problems, in plate deflection theory and thus plays a pivotal role in several branches of engineering and sciences. On delving through the literature, we found that analytical and numerical treatment of these equations have drawn a lot of attention of many researchers. Various two- and three- level numerical schemes for the fourth order parabolic PDEs without third order derivative terms were studied by Crandall [1-2], Conte [3], Evans [4], Fairweather and Gourley [5], Andrade and McKee [6], and Jain et al. [7]. Using multi-derivative method, Twizell and Khaliq [8] proposed a stable difference scheme with high accuracy in time for the fourth order parabolic equations with constant coefficients. Later, Khaliq and Twizell [9] developed a family of second order methods based on three-point recurrence relation for the fourth order parabolic equations with variable coefficients. Evans and Yousif [10] solved the fourth order parabolic equation governing the transverse vibrations of a homogenous beam by using the alternating group explicit iteration method. Using Adomian decomposition method, Wazwaz [11-12] obtained analytic solution for fourth order linear parabolic equation. Christov et al. [13] used implicit time-splitting technique to solve fourth order parabolic equation. Richardson extrapolation technique is used by Liao et al. [14] on the computed solution to increase the order of accuracy for the solution of the system of reaction-diffusion equations. In [15-16], Mohanty and Evans propounded two- and three- level implicit methods for the solution of fourth order quasilinear parabolic equations without third order mixed derivative term. Aziz et al. [17] used spline approximations to obtain the numerical solution of transverse vibrations. Later, Biazari and Ghazvini [18] applied He's variational iteration method for fourth order parabolic PDEs with variable coefficients. Caglar and Caglar [19] used fifth degree B-spline approximations to solve fourth order parabolic equations. Dehghan and Manafian [20] achieved analytical solution of fourth order parabolic PDEs by applying He's homotopy perturbation method. Liao and Khaliq [21] developed an efficient fourth order numerical algorithm for solving a convection-diffusion equation by converting the original problem to a coupled system of two diffusion equations by introducing a new unknown function. Noor et al. [22] employed variational iteration technique for solving singular fourth order parabolic equations. Rashidinia and Mohammadi [23] proposed three-level implicit schemes by applying sextic spline functions for obtaining smooth approximations for the solution of fourth order variable coefficient, non-homogenous parabolic PDE dealing with vibration of beams. Most recently, Mohammadi [24] developed a numerical method based on sextic B-spline for the fourth order time dependent PDEs subjected to fixed and cantilever boundary conditions arising in robotic designs.

In the present note, we reduce the fourth-order parabolic PDE (1) into an equivalent form of two second order differential equations and transform the initial and boundary conditions accordingly. Using three spatial grid points, we discuss two-level implicit methods of Crank-Nicolson type and Crandall's type for the solution of differential Eq. (1). These methods are unconditionally stable and we do not require to discretize the boundary conditions. The simplicity of the proposed methods lies in their three point discretization without any use of fictitious points for incorporating the boundary conditions. The proposed Crandall's type method is fourth order accurate in space when the mesh size in time direction is directly proportional to the square of mesh size in space direction and thus provides much more accurate results. The numerical solution of the one-dimensional time-dependent Laplacian $u_{xx}-u_t$, which is quite often of interest in many applied problems is computed as by-product of the methods. In the next section, we present the implicit finite difference schemes. In Section 3, stability analysis of the proposed methods has been discussed theoretically. Test examples are considered to illustrate the usefulness of the developed methods and their theoretical behaviours are confirmed in Section 4 while Section 5 is devoted to concluding remarks.

2. Numerical methods

We cover the solution domain $\Omega=\{(x, t)|0 < x < 1, t > 0\}$ by a rectangular grid with spacing $\Delta x > 0$ and $\Delta t > 0$ in space and time dimensions, respectively. We replace the solution domain by a set of grid points $(x_l,t_j)$where $x_l=l\Delta x$, $t_j=j\Delta t$, $l=0, 1, \dots, N+1$, $\Delta x=1/(N+1)$, N is a positive integer and $j=0, 1, 2, \dots$. The mesh ratio parameter is denoted by $\lambda=(\Delta t/\Delta x^2)$. Let $U_l^j = u(x_l, t_j)$ be the exact solution of $u(x,t)$ at the grid point (x_l,t_j). We approximate the solution of Eq. (1) at the grid point (x_l,t_j) by u_l^j.

To transform the problem (1) into a system of two second order equations, we introduce a new variable $v(x, t)=u_{xx}(x,t)-u_t(x,t)$. Note that, for $u(x, t)$, the initial and Dirichlet boundary conditions are given by $u(x, 0)=\phi(x)$, $u(0, t)=b_0(t)$ and $u(1, t)=b_1(t)$, respectively. Since the grid lines are parallel to the coordinate axes and the value of u is exactly known on the boundary, this implies, the successive tangential partial derivatives of u are known on the boundary. Since $\phi(x)$ is assumed to be smooth enough, we can derive the initial condition for $u_{xx}(x,t)$ at $t=0$ by differentiating $\phi(x)$ twice with respect to x, so $u_{xx}(x,0)=\phi''(x)$. Likewise, $u_t(0,t)=b'_0(t)$ and $u_t(1,t)=b'_1(t)$. Hence the values of $u_{xx}(x,0)-u_t(x,0)$, $u_{xx}(0,t)-u_t(0,t)$ and $u_{xx}(1,t)-u_t(1,t)$ are known exactly.

Then, the PDE (1) can be reformulated as

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + v, \ \ 0 < x < 1, t > 0 \tag{4.1}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} + f(x, t, u), \ \ 0 < x < 1, t > 0 \tag{4.2}$$

The reformulated problem (4.1,4.2) is associated with the following initial and boundary conditions:

$$u(x,0)=\phi(x),\quad v(x,0)=\phi''(x)-\psi(x),\quad 0\le x\le 1 \tag{5}$$

$$u(0,t)=b_0(t),\quad v(0,t)=c_0(t)-b_0'(t),\quad u(1,t)=b_1(t),\quad v(1,t)=c_1(t)-b_1'(t),\quad t>0 \tag{6}$$

In this paper, we obtain two sets of finite difference methods. CASE-I: Difference method of order two in time and two in space requiring only one evaluation of the function *f*. CASE-II: Difference method of order two in time and four in space requiring three evaluations of the function *f*. For both the methods, we use only three spatial grid points.

At the grid point (x_l,t_j), the differential Eqs. (4.1) and (4.2) may be written as

$$\frac{\partial^2 U_l^j}{\partial x^2}=\frac{\partial U_l^j}{\partial t}+V_l^j, \tag{7.1}$$

$$\frac{\partial^2 V_l^j}{\partial x^2}=\frac{\partial V_l^j}{\partial t}+f(x_l,t_j,U_l^j)\equiv F_l^j. \tag{7.2}$$

Differentiating the Eqs. (4.1) and (4.2) with respect to '*t*' at the grid point (x_l,t_j), we obtain relations of the form

$$\frac{\partial^3 U_l^j}{\partial x^2\partial t}=\frac{\partial^2 U_l^j}{\partial t^2}+\frac{\partial V_l^j}{\partial t}, \tag{8.1}$$

$$\frac{\partial^3 V_l^j}{\partial x^2\partial t}=\frac{\partial^2 V_l^j}{\partial t^2}+G+H\frac{\partial U_l^j}{\partial t}, \tag{8.2}$$

where at the grid point (x_l,t_j), we denote $G=\frac{\partial f}{\partial t}$ and $H=\frac{\partial f}{\partial U}$.

In both the cases, we require the following approximations:

For $r=0,\ \pm 1$, we denote

$$\bar{t}_j=t_j+\theta\,\Delta t,\quad 0<\theta<1, \tag{9.1}$$

$$\overline{U}_{l+r}^j=\theta U_{l+r}^{j+1}+(1-\theta)U_{l+r}^j, \tag{9.2}$$

$$\overline{V}_{l+r}^j=\theta V_{l+r}^{j+1}+(1-\theta)V_{l+r}^j, \tag{9.3}$$

$$\overline{U}_{t_{l+r}}^j=(U_{l+r}^{j+1}-U_{l+r}^j)/\Delta t, \tag{9.4}$$

$$\overline{V}_{t_{l+r}}^j=(V_{l+r}^{j+1}-V_{l+r}^j)/\Delta t. \tag{9.5}$$

Using the above approximations, we define

$$\overline{F}_{l+r}^j=f(x_{l+r},\bar{t}_j,\overline{U}_{l+r}^j)\cong\bar{f}_{l+r}^j=f(x_{l+r},\bar{t}_j,\theta u_{l+r}^{j+1}+(1-\theta)u_{l+r}^j). \tag{10}$$

Method –I: A two-level implicit method of Crank-Nicolson type for the system of differential Eqs. (4.1) and (4.2) may be written as

$$\lambda\delta_x^2\overline{U}_l^j=\Delta t(\overline{U}_{t_l}^j+\overline{V}_l^j)+T_{11}, \tag{11.1}$$

$$\lambda{\delta_x}^2\overline{V}_l^j=\Delta t(\overline{V}_{t_l}^j+\overline{F}_l^j)+T_{12},\ l=1,\dots,N,\quad j=0,1,2,\dots \tag{11.2}$$

where δ_x is the central difference operator in *x*-direction defined by $\delta_x U_l=U_{l+\frac12}-U_{l-\frac12}$, and T_{11}, T_{12} are the local truncation errors associated with (11.1) and (11.2), respectively.

Expanding each term in Taylor series about (x_l,t_j) in (11.1) and (11.2) and simplifying with the aid of relations (7.1,7.2) and (8.1,8.2), we get

$$T_{11}=\Delta t^2\left(\theta-\frac12\right)\frac{\partial^2 U_l^j}{\partial^2 t}+O(\Delta t^3+\Delta t\Delta x^2) \tag{12.1}$$

$$T_{12}=\Delta t^2\left(\theta-\frac12\right)\frac{\partial^2 V_l^j}{\partial^2 t}+O(\Delta t^3+\Delta t\Delta x^2) \tag{12.2}$$

Note that we need to determine the parameter θ so that truncation errors T_{11} and T_{12} are of $O(\Delta t^3+\Delta t\,\Delta x^2)$. The proposed difference method (11.1) and (11.2) to be of $O(\Delta t^2+\Delta x^2)$, the coefficient of Δt^2 in (12.1) and (12.2) must be zero. Hence we obtain $\theta=1/2$ and the local truncation errors reduce to $T_{11}=O(\Delta t^3+\Delta t\Delta x^2)$ and $T_{12}=O(\Delta t^3+\Delta t\Delta x^2)$.

Neglecting the error terms in (11.1) and (11.2) and taking $\theta=1/2$, we may obtain the scheme for the system of Eqs. (4.1) and (4.2) as

$$\frac{\lambda}{2}{\delta_x}^2(u_l^{j+1}+u_l^j)=(u_l^{j+1}-u_l^j)+\frac{\Delta t}{2}(v_l^{j+1}+v_l^j), \tag{13.1}$$

$$\frac{\lambda}{2}\delta_x^{\ 2}(v_l^{j+1}+v_l^j)=(v_l^{j+1}-v_l^j)+\Delta t\overline{f}_l^j,\ l=1,\ldots,N,\quad j=0,1,2,\ldots \tag{13.2}$$

Method –II: A two-level implicit method of Crandall's type for the system of differential Eqs. (4.1) and (4.2) may be written as

$$12\lambda\delta_x^{\ 2}\overline{U}_l^j=\Delta t[(\overline{U}_{t_{l+1}}^{\ j}+\overline{V}_{l+1}^j)+10(\overline{U}_{t_l}^{\ j}+\overline{V}_l^j)+(\overline{U}_{t_{l-1}}^{\ j}+\overline{V}_{l-1}^j)]+T_{21}, \tag{14.1}$$

$$12\lambda\delta_x^{\ 2}\overline{V}_l^j=\Delta t[(\overline{V}_{t_{l+1}}^{\ j}+\overline{F}_{l+1}^j)+10(\overline{V}_{t_l}^{\ j}+\overline{F}_l^j)+(\overline{V}_{t_{l-1}}^{\ j}+\overline{F}_{l-1}^j)]+T_{22},\ l=1,\ldots,N,\ j=0,1,2,\ldots \tag{14.2}$$

where T_{21} and T_{22} are the local truncation errors associated with (14.1) and (14.2), respectively. In this case, we have to find the value of the parameter θ so that local truncation errors T_{21} and T_{22} are of $O(\Delta t^3+\Delta t\,\Delta x^4)$.

Using the approximations (9.1)-(9.5) and expanding each term about (x_l,t_j) in (14.1) and (14.2) by the help of Taylor series expansion, and then applying the relations (8.1) and (8.2) yields

$$T_{21}=\Delta t\,\Delta x^2\left(\frac{\partial^4U_l^j}{\partial x^4}-\frac{\partial^3U_l^j}{\partial x^2\partial t}-\frac{\partial^2V_l^j}{\partial x^2}\right)+\Delta t^2\left(\theta-\frac{1}{2}\right)\frac{\partial^2U_l^j}{\partial^2t}+O(\Delta t^3+\Delta t\Delta x^4) \tag{15.1}$$

$$T_{22}=\Delta t\,\Delta x^2\left(\frac{\partial^4V_l^j}{\partial x^4}-\frac{\partial^3V_l^j}{\partial x^2\partial t}-\frac{\partial^2F_l^j}{\partial x^2}\right)+\Delta t^2\left(\theta-\frac{1}{2}\right)\frac{\partial^2V_l^j}{\partial^2t}+O(\Delta t^3+\Delta t\Delta x^4) \tag{15.2}$$

Differentiating the Eqs. (4.1) and (4.2) twice successively with respect to 'x' at the grid point (x_l,t_j), we obtain relations of the form

$$\frac{\partial^4U_l^j}{\partial x^4}=\frac{\partial^3U_l^j}{\partial x^2\partial t}+\frac{\partial^2V_l^j}{\partial x^2}, \tag{16.1}$$

$$\frac{\partial^4V_l^j}{\partial x^4}=\frac{\partial^3V_l^j}{\partial x^2\partial t}+\frac{\partial^2F_l^j}{\partial x^2} \tag{16.2}$$

Using (16.1) and (16.2) in (15.1) and (15.2), the local truncation errors become

$$T_{21}=\Delta t^2\left(\theta-\frac{1}{2}\right)\frac{\partial^2U_l^j}{\partial^2t}+O(\Delta t^3+\Delta t\Delta x^4) \tag{17.1}$$

$$T_{22}=\Delta t^2\left(\theta-\frac{1}{2}\right)\frac{\partial^2V_l^j}{\partial^2t}+O(\Delta t^3+\Delta t\Delta x^4) \tag{17.2}$$

The proposed difference method (14.1) and (14.2) to be of $O(\Delta t^2+\Delta x^4)$, the coefficient of Δt^2 in (17.1) and (17.2) must be zero. Thus we obtain $\theta=1/2$ and the local truncation errors reduce to $T_{21}=O(\Delta t^3+\Delta t\Delta x^4)$ and $T_{22}=O(\Delta t^3+\Delta t\Delta x^4)$. For a fixed value of λ, the local truncation errors become $T_{21}=O(\Delta x^6)$ and $T_{22}=O(\Delta x^6)$.

Neglecting error terms in (14.1) and (14.2) and for $\theta=1/2$, we may obtain the scheme for the system of Eqs. (4.1) and (4.2) as

$$6\lambda\delta_x^{\ 2}(u_l^{j+1}+u_l^j)=(12+\delta_x^{\ 2})(u_l^{j+1}-u_l^j)+\frac{\Delta t}{2}(12+\delta_x^{\ 2})(v_l^{j+1}+v_l^j), \tag{18.1}$$

$$6\lambda\delta_x^{\ 2}(v_l^{j+1}+v_l^j)=(12+\delta_x^{\ 2})(v_l^{j+1}-v_l^j)+\Delta t(\overline{f}_{l+1}^j+\overline{f}_{l-1}^j+10\overline{f}_l^j),\ l=1,\ldots,N,\ j=0,1,2,\ldots \tag{18.2}$$

The order of the Crandall's type method (18.1) and (18.2) is defined as $(\Delta t)^{-1}O(\Delta t^3+\Delta t\,\Delta x^4)=O(\Delta t^2+\Delta x^4)$. It is fourth order accurate in space but only second order accurate in time, hence for a fixed value of the mesh ratio parameter λ, it is fourth order accurate in space.

3. Stability analysis

In this section, we discuss the stability of the methods presented earlier. We consider the linear fourth order parabolic equation of the form

$$\frac{\partial^4u}{\partial x^4}-2\frac{\partial^3u}{\partial x^2\partial t}+\frac{\partial^2u}{\partial t^2}=f(x,t),\ 0<x<1,t>0 \tag{19}$$

Applying the methods (13.1,13.2) and (18.1,18.2) to the differential Eq. (19), we obtain the matrix equation

$$\boldsymbol{Aw}^{\boldsymbol{j+1}}=\boldsymbol{Bw}^{\boldsymbol{j}}+\boldsymbol{d} \tag{20}$$

where

$$A=\begin{bmatrix}\boldsymbol{A_1} & \boldsymbol{A_2}\\ \boldsymbol{0} & \boldsymbol{A_1}\end{bmatrix},\quad \boldsymbol{B}=\begin{bmatrix}\boldsymbol{B_1} & \boldsymbol{B_2}\\ \boldsymbol{0} & \boldsymbol{B_1}\end{bmatrix},\ \boldsymbol{w}=\begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{bmatrix}\text{ and }\boldsymbol{d}=\begin{bmatrix}\boldsymbol{d_1}\\ \boldsymbol{d_2}\end{bmatrix}.$$

The submatrices for *Method*-I are given by $\boldsymbol{A}_1=[0,1,0]-\frac{\lambda}{2}[1,-2,1]$, $\boldsymbol{A}_2=\frac{\Delta t}{2}[0,1,0]$, $\boldsymbol{B}_1=[0,1,0]+\frac{\lambda}{2}[1,-2,1]$, $\boldsymbol{B}_2=\frac{-\Delta t}{2}[0,1,0]$, and the submatrices for *Method*-II are given by $\boldsymbol{A}_1=[1,10,1]-6\lambda\ [1,-2,1]$, $\boldsymbol{A}_2=\frac{\Delta t}{2}[1,10,1]$, $\boldsymbol{B}_1=[1,10,1]+\ 6\lambda\ [1,-2,1]$, $\boldsymbol{B}_2=\frac{-\Delta t}{2}[1,10,1]$, where $[a,b,c]$ denotes the N^{th}-order tri-diagonal matrix whose eigenvalues are given by $b+2\sqrt{ac}\cos(2\phi)$, $2\varphi=s\pi/(N+1)$, $s=1,\ \ldots\ ,\ N$; $\boldsymbol{u}$, $\boldsymbol{v}$ are solution vectors and $\boldsymbol{d}_1$, $\boldsymbol{d}_2$ are vectors consisting of homogeneous functions, initial and boundary values associated with the block system given by (20).

The eigenvalues of submatrices $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ for *Method*-I are given by $1+\ 2\lambda\sin^2\phi$, $\frac{\Delta t}{2}$, $1-2\lambda\sin^2\phi$ and $\frac{-\Delta t}{2}$, respectively, and hence the eigenvalues of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ for *Method*-I are given by $1+2\lambda\sin^2\phi$ and $1-2\lambda\sin^2\phi$, respectively.

The eigenvalues of submatrices $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ for *Method*-II are given by $12-4\sin^2\phi+24\lambda\sin^2\phi$, $\frac{\Delta t}{2}[12-4\sin^2\phi]$, $12-4\sin^2\phi-24\lambda\sin^2\phi$ and $\frac{-\Delta t}{2}[12-4\sin^2\phi]$, respectively, and hence the eigenvalues of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ for *Method*-II are given by $12-4\sin^2\phi+24\lambda\sin^2\phi$ and$12-4\sin^2\phi-24\lambda\sin^2\phi$, respectively.

The amplification matrix of the system (20) is given by $\boldsymbol{A}^{-1}\boldsymbol{B}$. Since the matrices $\boldsymbol{A}^{-1}$and $\boldsymbol{B}$ commute each other, hence eigenvalues ξ of $\boldsymbol{A}^{-1}\boldsymbol{B}$ are given by

$$\xi=\frac{1-2\lambda\sin^2\phi}{1+2\lambda\sin^2\phi},\ \text{(for } Method-\text{I)} \tag{21}$$

and

$$\xi=\frac{12-4\sin^2\phi-24\lambda\sin^2\phi}{12-4\sin^2\phi+24\lambda\sin^2\phi},\ \text{(for } Method-\text{II)} \tag{22}$$

Since $0\le\sin^2\phi\le1$, from (21) and (22)), it is easy to verify that $|\xi|\le1$ for all variable angle ϕ and $\lambda>0$. Hence the methods are unconditionally stable.

We see that $\lim_{(\Delta x,\Delta t)\to(0,0)}(\Delta t)^{-1}T_{11}=0$, $\lim_{(\Delta x,\Delta t)\to(0,0)}(\Delta t)^{-1}T_{12}=0$, $\lim_{(\Delta x,\Delta t)\to(0,0)}(\Delta t)^{-1}T_{21}=0$ and $\lim_{(\Delta x,\Delta t)\to(0,0)}(\Delta t)^{-1}T_{22}=0$ and therefore the proposed difference methods (13.1,13.2) and (18.1,18.2) are consistent with the differential Eqs. (4.1) and (4.2). Hence, by *Lax Equivalence Theorem* [27], the methods are convergent.

4. Computational results

In this section, we implement the proposed methods over the following *two* test problems whose exact solutions are known. The right hand side homogeneous functions, the initial and boundary conditions are determined using the exact solutions as a test procedure. The coupled system of equations are solved using iterative methods. The linear difference equations have been solved by block Gauss-Seidel iteration method whereas the non-linear difference equations have been solved using the block Newton's iterative method for nonlinear problems (see [25-27]). In all cases, we have selected $\boldsymbol{u}^{(0)}=\boldsymbol{0}$ as the initial guess and the iterations were terminated when the absolute error tolerance, $|\boldsymbol{u}^{(s+1)}-\boldsymbol{u}^{(s)}|\le10^{-15}$ was achieved. All the computations were carried out using MATLAB codes. The order of accuracy for the solution variable u is obtained using the formula

$$\frac{\log(e_{h_1}/e_{h_2})}{\log(h_1/h_2)}, \tag{23}$$

where e_{h_1} and e_{h_2} are the maximum absolute errors in u for two uniform mesh sizes h_1 and h_2, respectively,

Example 1. The partial differential Eq. (19) is to be solved whose exact solution is given by $u(x,t)=e^{-t}\sin(\pi x)$. The maximum absolute errors, cpu time (in seconds), and order of convergence at $t=1$ and $t=2$ are reported in Table 1 for a fixed value of $\lambda=1.6$. The plots of numerical solution obtained using the method (18.1) and (18.2) and the exact solution are given in Fig. 1a and Fig. 1b, respectively.

Example 2.

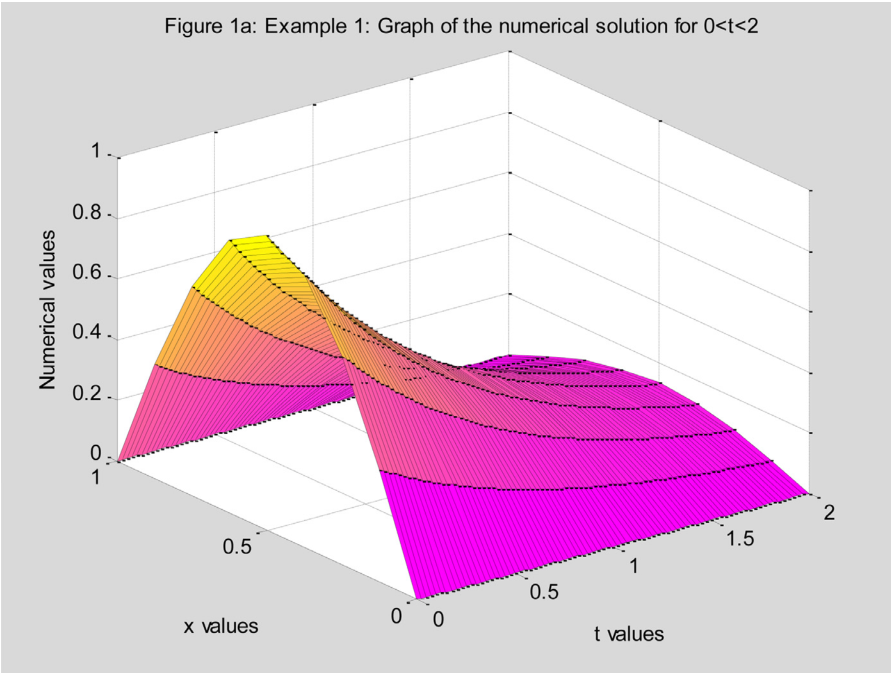
$$\frac{\partial^4u}{\partial x^4}-2\frac{\partial^3u}{\partial x^2\partial t}+\frac{\partial^2u}{\partial t^2}=1+u^2+f(x,t),\ \ 0<x<1,t>0 \tag{24}$$

The exact solution is given by $u(x,\ t)=\cosh x\sin t$. The maximum absolute errors, cpu time (in seconds), and numerical order of convergence at $t=5$ and $t=10$ are reported in Table 2 for a fixed value of $\lambda=1.6$. The graphs of numerical solution using method (18.1) and (18.2) and exact solution are plotted in Fig. 2a and Fig. 2b, respectively.

It is evident from Tables 1 and 2 that the computational results are in good agreement with the exact solutions and the order of convergence is numerically estimated to be equal to four in case of Crandall's type method (18.1,18.2) for a fixed value of the mesh ratio parameter. This confirms theoretical order of convergence established in Section 2.

5. Concluding remarks

In this paper, using three spatial grid points, we discuss two-level unconditionally stable implicit methods of order two in time and two in space of Crank-Nicolson type and two in time and four in space of Crandall's type for the numerical solution of fourth order mildly nonlinear parabolic equations. The performance of the algorithms has been tested over the considered



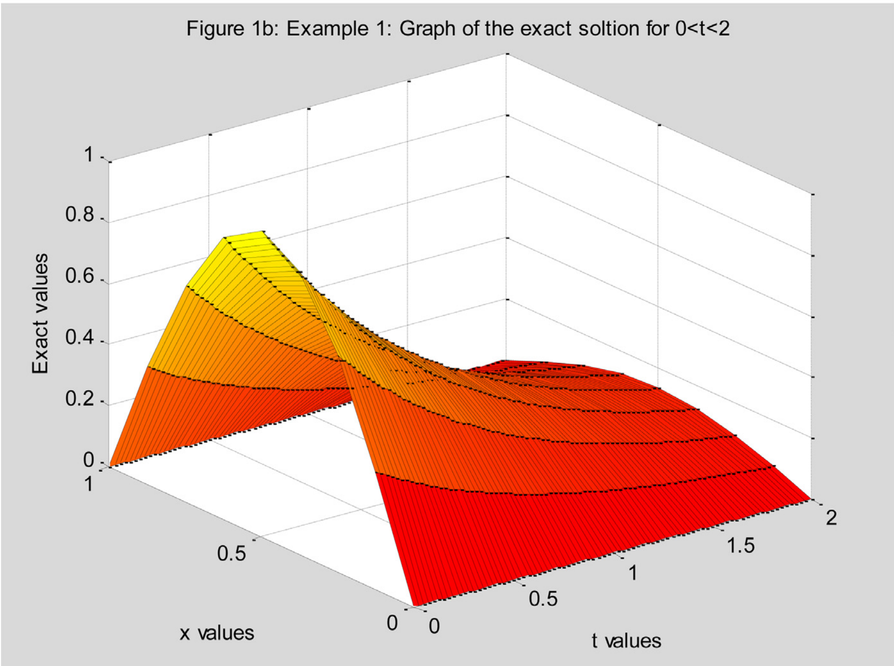
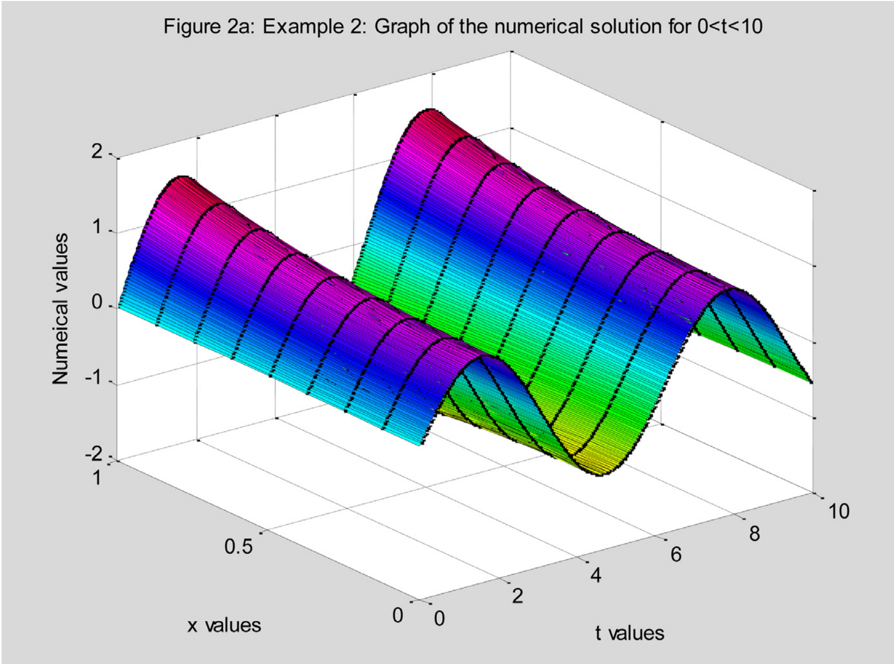


Fig. 1. (a): Example 1: The graph of numerical solution for $\Delta x = 1/8$, $\lambda = 1.6$, $0 < t < 2$. (b): Example 1: The graph of exact solution for $\Delta x = 1/8$, $\lambda = 1.6$, $0 < t < 2$.

test problems and maximum absolute errors have been computed. It is found that the simulating results approximate the analytical solutions very well. More precisely, we have the following main outcomes:

1. *Compact schemes*: Use of three spatial grid points makes the methods more efficient and computationally economical. Ease of implementation and better accuracy is achieved in considerably smaller number of nodes.
2. *No fictitious nodes*: We do not require to discretize the boundary conditions and no fictitious points outside the solution region are needed for incorporating the boundary conditions.



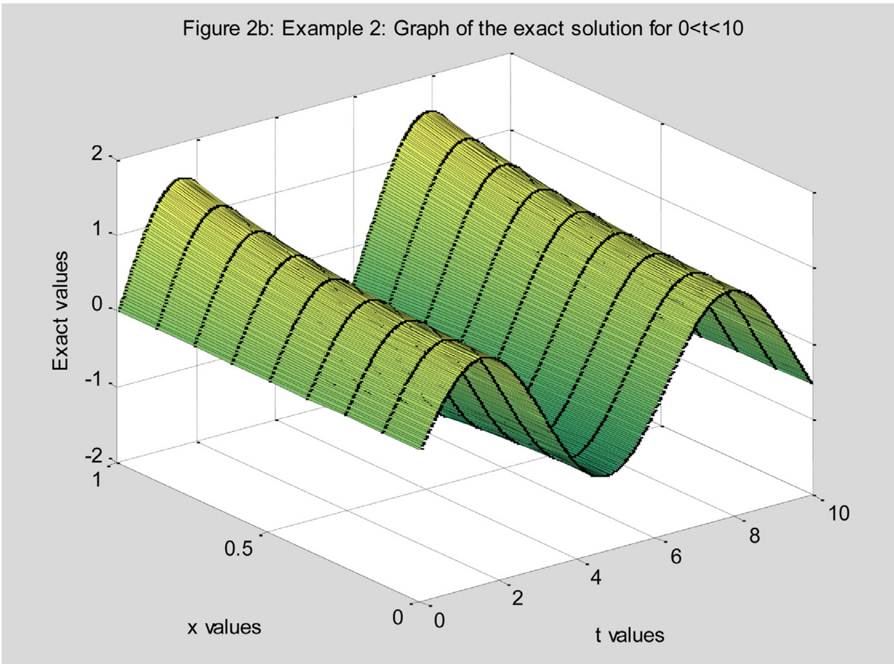


Fig. 2. (a): Example 2: The graph of numerical solution for $\Delta x = 1/8$, $\lambda = 1.6$, $0 < t < 10$. (b): Example 2: The graph of exact solution for $\Delta x = 1/8$, $\lambda = 1.6$, $0 < t < 10$.

3. *High accuracy*: The proposed Crandall's type method (18.1) and (18.2) provides fourth order convergent results for a fixed mesh ratio parameter λ.
4. *Unconditional stability*: The unconditional stability of the proposed methods allows the use of much large time step which is useful when the problem is solved on a long time interval.

The authors are currently working to extend this splitting technique to solve fourth order parabolic equations in two and three space dimensions.

Table 1
Example 1: The maximum absolute errors.

Δx	Method (13.1) and (13.2)		Method (18.1) and (18.2)	
	$t=1$	$t=2$	$t=1$	$t=2$
1/8 u	1.0652(−02)	3.9220(−03)	4.8546(−05)	1.7866(−05)
v	4.6805(−02)	1.7221(−02)	8.7892(−05)	3.2338(−05)
cpu time	(0.0245)	(0.0264)	(0.0250)	(0.0351)
1/16 u	2.6369(−03)	9.7083(−04)	3.0106(−06)	1.1080(−06)
v	1.1673(−02)	4.2948(−03)	5.3899(−06)	1.9831(−06)
cpu time	(0.0613)	(0.1103)	(0.0720)	(0.1278)
order	2.0142	2.0143	4.0112	4.0112
1/32 u	6.5762(−04)	2.4211(−04)	1.8780(−07)	6.9115(−08)
v	2.9164(−03)	1.0730(−03)	3.3525(−07)	1.2335(−07)
cpu time	(0.3659)	(1.1961)	(0.3536)	(0.7137)
order	2.0035	2.0036	4.0028	4.0028
1/64 u	1.6430(−04)	6.0490(−05)	1.1732(−08)	4.3175(−09)
v	7.2900(−04)	2.6822(−04)	2.0928(−08)	7.6999(−09)
cpu time	(2.6623)	(6.9642)	(2.5067)	(4.3647)
order	2.0009	2.0009	4.0007	4.0007

Table 2
Example 2: The maximum absolute errors.

Δx	Method (13.1) and (13.2)		Method (18.1) and (18.2)	
	$t=5$	$t=10$	$t=5$	$t=10$
1/8 u	1.6286(−04)	8.7519(−05)	4.7348(−06)	7.4039(−06)
v	2.2135(−04)	6.9080(−05)	3.4358(−05)	3.5696(−06)
cpu time	(0.0618)	(0.0934)	(0.0485)	(0.0852)
1/16 u	3.9876(−05)	2.3277(−05)	2.9590(−07)	4.6269(−07)
v	6.1718(−05)	1.7901(−05)	2.1502(−06)	2.2290(−07)
cpu time	(0.3185)	(0.6542)	(0.2650)	(0.5259)
order	2.0300	1.9107	4.0001	4.0002
1/32 u	9.9317(−06)	5.9134(−06)	1.8494(−08)	2.8921(−08)
v	1.5847(−05)	4.5142(−06)	1.3480(−09)	1.3981(−08)
cpu time	(2.3359)	(4.7619)	(1.7863)	(3.6016)
order	2.0054	1.9768	4.0000	3.9999
1/64 u	2.4801(−06)	1.4843(−06)	1.1564(−09)	1.8092(−09)
v	3.9886(−06)	1.1317(−06)	8.4244(−09)	8.7374(−10)
cpu time	(17.0309)	(36.9899)	(11.8196)	(25.3948)
order	2.0016	1.9942	3.9993	3.9987

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