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Two-level implicit high order method based on half-step discretization for 1D unsteady biharmonic problems of first kind

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ABSTRACT

In this work, compact difference scheme based on half-step discretization is proposed to solve fourth order time dependent partial differential equations subject to Dirichlet and Neumann boundary conditions. The difference method reported here is second order accurate in time and fourth order accurate in space. The scheme employs only three grid points at each time-level and the given boundary conditions are exactly satisfied with no further approximations at the boundaries. The linear stability of the presented method has been examined and it is shown that the proposed two-level finite difference method is unconditionally stable for a model linear problem. The developed method is directly applicable to fourth order parabolic partial differential equations with singular coefficients which is the main highlight of our work. The method is successfully tested on singular problem. The proposed method is applied to find the numerical solution of the classical nonlinear Kuramoto–Sivashinsky equation and the extended Fisher–Kolmogorov equation. Comparison of the obtained results with those of some earlier known methods demonstrate the superiority of the present approach.

1. Introduction

In the present paper, we consider the class of fourth order parabolic partial differential equations (PDEs) of the form:

$$A(x,t)\frac{\partial^4 u}{\partial x^4} + \frac{\partial u}{\partial t} = f(x,t,u,u_x,u_{xx},u_{xxx}), \quad (x,t) \in \Omega \tag{1}$$

subject to the initial condition

$$u(x,0) = u_0(x), \quad a \le x \le b, \tag{2}$$

alongwith Dirichlet and Neumann boundary conditions of the type:

$$u(a,t) = g_0(t), \quad u(b,t) = g_1(t), \quad t > 0, \tag{3.1}$$

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$$\frac{\partial u}{\partial x}(a,t)=h_0(t), \quad \frac{\partial u}{\partial x}(b,t)=h_1(t), \quad t>0, \tag{3.2}$$

where $\Omega \equiv \{(x,t)| a<x<b, t>0\}$ and f, u_0, g_0, g_1, h_0, h_1 are functions of sufficient smoothness. It is assumed that the solution $u(x,t)$ is smooth enough to maintain the order and accuracy of the difference scheme under consideration. The differential equation (1) subject to the initial condition (2) and with u, $\partial u/\partial n$ prescribed at the boundary is called the unsteady biharmonic problem of first kind.

The class of PDEs (1) is a generalization of numerous significant mathematical physics equations. It includes, in particular, the generalized Kuramoto–Sivashinsky (GKS) equation arising in the study of continuous media exhibiting a chaotic behavior form:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\alpha\frac{\partial^2 u}{\partial x^2}+\beta\frac{\partial^3 u}{\partial x^3}+\gamma\frac{\partial^4 u}{\partial x^4}=0, \quad (x,t)\in\Omega, \tag{4}$$

where α, β and γ are real constants. The GKS equation is also known as KdV–Burgers–Kuramoto equation and plays an essential part in depicting physical processes in unstable systems [20,32]. Equation (4) has been employed to explain nonlinear waves, flame front propagation and unstable drift waves in plasma [19,31,32].

For $\beta=0$, equation (4) is known as the Kuramoto–Sivashinsky (KS) equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\alpha\frac{\partial^2 u}{\partial x^2}+\gamma\frac{\partial^4 u}{\partial x^4}=0, \quad (x,t)\in\Omega, \tag{5}$$

where α and γ are constants associated with the growth of linear stability and surface tension, respectively. The KS equation is a fundamental fourth order PDE modeling various nonlinear physical phenomenon in unstable systems and reveals complex chaotic behavior often encountered in the study of continuous media. It arises in a wide range of physical processes such as in representing long waves on the interface between two viscous fluids [16], in flame propagation [32], in reaction–diffusion combustion dynamics and in the study of unstable drift waves in plasmas [20]. In this framework, it was introduced independently by Kuramoto and Tsuzuki [20] in order to study dissipative structure of reaction–diffusion and by Sivashinsky [32] as a model for analyzing flame front propagation during mild combustion processes and instabilities induced by thermal conduction of the considered gas. The KS equation comprises of high order dissipation term u_{xxxx}, nonlinear advection uu_x and linear growth term u_{xx}. Thus, in general the KS equation comprises a nonlinear initial valued problem concerning fourth order spatial derivative. When $\gamma=0$, the surface tension term is eliminated and the KS equation reduces to Burger's equation. It has emerged as a primary evolution equation for explaining highly nonlinear physical phenomenon in unstable systems.

The extended Fisher–Kolmogorov (EFK) equation is another significant time-dependent fourth order PDE:

$$\frac{\partial u}{\partial t}+\gamma\frac{\partial^4 u}{\partial x^4}-\frac{\partial^2 u}{\partial x^2}+f(u)=0 \tag{6}$$

where $f(u)=u^3-u$ and γ is a positive constant. When $\gamma=0$ in (6), it reduces to the standard Fisher–Kolmogorov equation. The EFK equation has got great significance as numerous problems in physics related to pattern formation and other bistable processes are mathematically modeled by it. When $\gamma>0$, Dee and van Saarloos [9] first proposed it as a higher order model equation arising in the study of pattern formation in bi-stable systems. It has a large variety of applications in the theory of propagation of domain walls in liquid crystals [37], instability in nematic liquid crystals [38] and traveling waves in reaction–diffusion systems [5]. The existence, uniqueness, regularity and other theoretical results for equation (6) have been discussed in [7].

The analytical treatment of these equations is too involved a process and do not appear to be very practical in most cases. Only limited class of specific PDEs can be solved analytically and it is not usually possible to determine closed-form solutions even in very simple cases. Therefore, numerical techniques for finding approximate solutions to these PDEs are sought after. In recent years, the solution of KS equation has been found by applying various methods including finite difference methods [1,2], tanh-function method [11], local discontinuous Galerkin method [35], Chebyshev spectral collocation method [18], radial basis function based on mesh free method [34], quintic B-spline collocation scheme [27] and lattice Boltzmann model [21]. Lakestani and Dehghan [22] designed a numerical procedure based on the finite difference and collocation methods for the solution of the GKS equation. Further, Haq et al. [15] developed a meshless method of lines for the GKS equation by approximating the spatial derivatives by radial basis functions using only a set of scattered nodes. Ye et al. [36] proposed a lattice Boltzmann model based on the higher-order moment method possessing fourth-order accuracy of truncation errors. Recently, Ganaie et al. [13] employed cubic Hermite functions to develop collocation method for the numerical solution of KS equation and derived a bound for maximum norm of the semi-discrete solution using Lyapunov functional. In [4], B-spline based finite element approach to the solution of the one-dimensional KS equation is presented. Most recently, Rashidinia and Jokar [29] developed a collocation method using the polynomial scaling functions for the numerical solution of the GKS equation. As far as computational studies for the EFK equation are concerned, Danumjaya and Pani [8] performed some numerical experiments to obtain approximate solution using orthogonal cubic spline collocation method. Doss and Nandini [10] presented a study on numerical implementation of the H^1-Galerkin mixed finite element cubic spline approximation method for the EFK equation by using a splitting technique. In recent years, there has been an increasing interest in the meshless numerical methods for simulating several important PDEs having physical relevance, see e.g., [12,23–26].

Compact finite difference method which is restricted to the patch of cells immediately surrounding any given mesh point is one of the attractive means for obtaining approximate solutions and are steadily gaining recognition due to the relative ease of implementation and flexibility. Apart from this, the advantage of developing a compact scheme is its suitability to be used directly adjacent to the boundary without introducing any extra nodes outside the boundary of the domain. With the solution and its first order space derivative prescribed at the boundary, the reduction of the fourth order equation to coupled second order equations is not possible and a high order compact approximation to the normal derivative has to be derived. Difficulties were experienced in the past for the higher order compact difference solution of fourth order parabolic equation in polar coordinates. The solution generally deteriorate in the vicinity of the singularity. Using three spatial grid points, Numerov type two level implicit difference method of order two in time and four in space for the solution of unsteady biharmonic problem of first kind was reported by Mohanty [28]. However, this method fails at singular points and require suitable modifications leading to tedious calculations to solve singular problems. The proposed method uses half-step points and is directly applicable to singular problems without any modification which is the foremost attraction of our work. The motivation comes from the article by Chawla and Shivakumar [6] who developed the fourth order half-step discretization for a two-point boundary value problem.

The approach involves discretizing the unsteady biharmonic equation not only using the values of the function u but also the values of the derivatives u_x at all internal grid points. It is required to solve the system of two equations to obtain the solutions of u and u_x at all grid points. In Section 2, we outline our new half-step discretization strategy for (1) and discuss the derivation of the scheme in detail. In Section 3, we apply the proposed method to a linear fourth order PDE in polar coordinates and carry out the von Neumann stability analysis for linear model problem. The performance of the proposed method is illustrated by numerical experiments done on a collection of test problems having physical significance including the highly nonlinear Kuramoto–Sivashinsky equation and the extended Fisher–Kolmogorov equation in Section 4. Conclusions are given in Section 5.

2. Two-level implicit compact difference scheme

In this paper, the procedure used to derive the finite difference method is new, although it is related with the work of Stephenson [33]. We define a new variable $v = u_x$, so the PDE (1) can be written as

$$A(x,t)\frac{\partial^4 u}{\partial x^4} + \frac{\partial u}{\partial t} = f(x,t,u,v,u_{xx},v_{xx}), \quad (x,t) \in \Omega \tag{7}$$

Our approach involves discretizing the unsteady biharmonic equation (7) using not only the values of the function u but also the values of the derivative $v = u_x$ at all internal grid points.

Let $h > 0$ and $k > 0$ be the mesh spacing in the space and time directions, respectively. Spatial knots are equally distributed over the solution domain $[a,b]$ as

$$a = x_0 < x_1 < \ldots < x_N < x_{N+1} = b,$$

with mesh spacing $h = (b-a)/(N+1)$ and $t_j = jk,\ j = 0,1,2,\ldots$, where N is a positive integer. Let $\lambda = (k/h^2) > 0$ be the mesh ratio parameter. We replace the region Ω by a set of grid points (x_l, t_j). Let the exact solution values of $u(x,t)$ and $v(x,t)$ at the grid point (x_l,t_j) be denoted by U_l^j and V_l^j, respectively and u_l^j and v_l^j denote their approximate solution values, respectively.

At the grid point (x_l,t_j), for $S = A, U$ and V we denote

$$S_{ab} = \frac{\partial^{a+b} S}{\partial x^a \partial t^b}, \ \ a,b = 0,1,\ldots.$$

Using the technique of Ananthakrishnaiah et al. [3] for developing high order methods for elliptic equations with variable coefficients on a uniform square grid by a procedure based on truncated Taylor series expansion of the solution and the coefficients of the governing equation, we first obtain

$$\begin{aligned}
&\left[A_l^j - \frac{2h^2}{15}\left(\frac{A_{x_l}^j}{A_l^j}\right) + \frac{h^2}{15}A_{xx_l}^j\right]\left[-2\left(U_{l+1}^j - 2U_l^j + U_{l-1}^j\right) + h\left(V_{l+1}^j - V_{l-1}^j\right)\right] \\
&+ \frac{h^4}{90}\left[\left(1 - h\frac{A_{x_l}^j}{A_l^j}\right)U_{t_{l+1}}^j + 13U_{t_l}^j + \left(1 + h\frac{A_{x_l}^j}{A_l^j}\right)U_{t_{l-1}}^j\right] \\
&= \frac{h^4}{90}\left[\left(4 - 2h\frac{A_{x_l}^j}{A_l^j}\right)F_{l+1/2}^j + 7F_l^j + \left(4 + 2h\frac{A_{x_l}^j}{A_l^j}\right)F_{l-1/2}^j\right] + O(h^8)
\end{aligned} \tag{8}$$

The above formula uses only three grid points and also contain the values of the coefficients at the grid points of the mesh.

For $r = 0, \pm 1$, we consider the following approximations:

$$\bar{t}_j = t_j + \theta k, \ \ 0 < \theta < 1, \tag{9.1}$$

$$\overline{A}_l^j = A(x_l, \bar{t}_j), \tag{9.2}$$

$$\overline{A}_{x_l}^j = A_x(x_l, \bar{t}_j), \tag{9.3}$$

$$\overline{A}_{xx_l}^j = A_{xx}(x_l, \bar{t}_j), \tag{9.4}$$

$$\overline{U}_{l+r}^j = \theta U_{l+r}^{j+1} + (1-\theta) U_{l+r}^j, \tag{9.5}$$

$$\overline{V}_{l+r}^j = \theta V_{l+r}^{j+1} + (1-\theta) V_{l+r}^j, \tag{9.6}$$

$$\overline{U}_{t_{l+r}}^j = (U_{l+r}^{j+1} - U_{l+r}^j)/k, \tag{9.7}$$

$$\overline{U}_{l+1/2}^j = \frac{1}{2}(\overline{U}_{l+1}^j + \overline{U}_l^j), \tag{9.8}$$

$$\overline{U}_{l-1/2}^j = \frac{1}{2}(\overline{U}_{l-1}^j + \overline{U}_l^j), \tag{9.9}$$

$$\overline{V}_{l+1/2}^j = \frac{1}{h}(\overline{U}_{l+1}^j - \overline{U}_l^j), \tag{9.10}$$

$$\overline{V}_{l-1/2}^j = \frac{1}{h}(\overline{U}_l^j - \overline{U}_{l-1}^j), \tag{9.11}$$

$$\overline{U}_{xx_l}^j = \frac{1}{h^2}(\overline{U}_{l+1}^j - 2\overline{U}_l^j + \overline{U}_{l-1}^j), \tag{9.12}$$

$$\overline{U}_{xx_{l+1/2}}^j = \frac{1}{h}(\overline{V}_{l+1}^j - \overline{V}_l^j), \tag{9.13}$$

$$\overline{U}_{xx_{l-1/2}}^j = \frac{1}{h}(\overline{V}_l^j - \overline{V}_{l-1}^j), \tag{9.14}$$

$$\overline{V}_{xx_l}^j = \frac{1}{h^2}(\overline{V}_{l+1}^j - 2\overline{V}_l^j + \overline{V}_{l-1}^j). \tag{9.15}$$

Using the approximations (9.1)–(9.15), we define

$$\overline{F}_l^j = f(x_l, \bar{t}_j, \overline{U}_l^j, \overline{V}_l^j, \overline{U}_{xx_l}^j, \overline{V}_{xx_l}^j) = F_l^j + O(k + h^2) \tag{10}$$

We need $O(k+h^2)$ approximation for v_{xx} at the half-step points $(x_{l\pm 1/2}, t_j)$. For this, let

$$h^3 \overline{V}_{xx_{l+1/2}}^j = a_0(\overline{U}_{l+1}^j - \overline{U}_{l-1}^j - 2h\overline{V}_l^j) + a_1(\overline{U}_{l+1}^j + \overline{U}_{l-1}^j - 2\overline{U}_l^j) + a_2 h(\overline{V}_{l+1}^j - \overline{V}_{l-1}^j) \tag{11}$$

where a_0, a_1 and a_2 are free parameters to be determined. Expanding each term on the right-hand side of (11) in Taylor series about (x_l, t_j) and equating the coefficients up to h^4 to zero, we obtain $(a_0, a_1, a_2) = (3, -6, 3)$. Thus, we obtain the approximation

$$\begin{aligned}\overline{V}_{xx_{l+1/2}}^j &= \frac{3}{h^3}(\overline{U}_{l+1}^j - \overline{U}_{l-1}^j - 2h\overline{V}_l^j) - \frac{6}{h^3}(\overline{U}_{l+1}^j + \overline{U}_{l-1}^j - 2\overline{U}_l^j) + \frac{3}{h^2}(\overline{V}_{l+1}^j - \overline{V}_{l-1}^j) \\ &= V_{xx_{l+1/2}}^j + O(k+h^2)\end{aligned} \tag{12}$$

Similarly, we have

$$\begin{aligned}\overline{V}_{xx_{l-1/2}}^j &= \frac{3}{h^3}(\overline{U}_{l+1}^j - \overline{U}_{l-1}^j - 2h\overline{V}_l^j) + \frac{6}{h^3}(\overline{U}_{l+1}^j + \overline{U}_{l-1}^j - 2\overline{U}_l^j) - \frac{3}{h^2}(\overline{V}_{l+1}^j - \overline{V}_{l-1}^j) \\ &= V_{xx_{l-1/2}}^j + O(k+h^2)\end{aligned} \tag{13}$$

Now, we define

$$\overline{F}_{l\pm 1/2}^j = f(x_{l\pm 1/2}, \bar{t}_j, \overline{U}_{l\pm 1/2}^j, \overline{V}_{l\pm 1/2}^j, \overline{U}_{xx_{l\pm 1/2}}^j, \overline{V}_{xx_{l\pm 1/2}}^j) = F_{l\pm 1/2}^j + O(k+h^2) \tag{14}$$

For the derivation of the high order method, we need an $O(k^2 + kh^2 + h^4)$ approximation for u_{xx} at (x_l, t_j). Suppose

$$h^2 \overline{\overline{U}}_{xx_l}^j = b_0(\overline{U}_{l+1}^j - 2\overline{U}_l^j + \overline{U}_{l-1}^j) + b_1 h(\overline{V}_{l+1}^j - \overline{V}_{l-1}^j) \tag{15}$$

where b_0 and b_1 are unknowns to be determined. Expanding each term on the right-hand side of (15) in Taylor series about (x_l, t_j) and equating the coefficients of h^2 and h^4 to zero, we obtain $(a_0, a_1) = (2, -\frac{1}{2})$. Hence the approximation

$$\overline{\overline{U}}^j_{xx_l} = \frac{2}{h^2}(\overline{U}^j_{l+1} - 2\overline{U}^j_l + \overline{U}^j_{l-1}) - \frac{1}{2h}(\overline{V}^j_{l+1} - \overline{V}^j_{l-1}) = U^j_{xx_l} + \theta k U_{21} + O(k^2 + kh^2 + h^4) \tag{16}$$

Likewise, we obtain

$$\overline{\overline{V}}^j_{xx_l} = \frac{15}{2h^3}(\overline{U}^j_{l+1} - \overline{U}^j_{l-1} - 2h\overline{V}^j_l) - \frac{3}{2h^2}(\overline{V}^j_{l+1} + \overline{V}^j_{l-1} - 2\overline{V}^j_l) = V^j_{xx_l} + \theta k U_{31} + O(k^2 + kh^2 + h^4) \tag{17}$$

Using (16)–(17), we define

$$\overline{\overline{F}}^j_l = f(x_l, \bar{t}_j, \overline{U}^j_l, \overline{V}^j_l, \overline{\overline{U}}^j_{xx_l}, \overline{\overline{V}}^j_{xx_l}). \tag{18}$$

Incorporating the approximations (16)–(17) in (18) and invoking Taylor series expansion, we find

$$\overline{\overline{F}}^j_l = F^j_l + \theta k T_0 + O(k^2 + kh^2 + h^4) \tag{19}$$

where $T_0 = \left(\frac{\partial f}{\partial t}\right)^j_l + U_{01}\left(\frac{\partial f}{\partial u}\right)^j_l + U_{11}\left(\frac{\partial f}{\partial v}\right)^j_l + U_{21}\left(\frac{\partial f}{\partial u_{xx}}\right)^j_l + U_{31}\left(\frac{\partial f}{\partial v_{xx}}\right)^j_l$.

Next, we consider the following additional high order approximations for the solution variables at the half-step points:

$$\overline{\overline{U}}^j_{l+1/2} = \frac{1}{2}(\overline{U}^j_{l+1} + \overline{U}^j_l) - \frac{h}{16}(\overline{V}^j_{l+1} - \overline{V}^j_{l-1}) = U^j_{l+1/2} + \theta k U_{01} + O(k^2 + hk + h^3), \tag{20.1}$$

$$\overline{\overline{U}}^j_{l-1/2} = \frac{1}{2}(\overline{U}^j_{l-1} + \overline{U}^j_l) - \frac{h}{16}(\overline{V}^j_{l+1} - \overline{V}^j_{l-1}) = U^j_{l-1/2} + \theta k U_{01} + O(k^2 + hk + h^3), \tag{20.2}$$

$$\overline{\overline{V}}^j_{l+1/2} = \frac{1}{2}(\overline{V}^j_{l+1} + \overline{V}^j_l) - \frac{1}{8}(\overline{V}^j_{l+1} - 2\overline{V}^j_l + \overline{V}^j_{l-1}) = V^j_{l+1/2} + \theta k U_{11} + O(k^2 + hk + h^3), \tag{20.3}$$

$$\overline{\overline{V}}^j_{l-1/2} = \frac{1}{2}(\overline{V}^j_{l-1} + \overline{V}^j_l) - \frac{1}{8}(\overline{V}^j_{l+1} - 2\overline{V}^j_l + \overline{V}^j_{l-1}) = V^j_{l-1/2} + \theta k U_{11} + O(k^2 + hk + h^3). \tag{20.4}$$

In order to achieve high order approximation for u_{xx} and v_{xx} at the half-step points, we consider the following linear combination:

$$h^2\overline{\overline{U}}^j_{xx_{l+1/2}} = c_0(\overline{U}^j_{l+1} - 2\overline{U}^j_l + \overline{U}^j_{l-1}) + c_1 h(\overline{V}^j_{l+1} - \overline{V}^j_{l-1}) + c_2(\overline{U}^j_{l+1} - \overline{U}^j_{l-1} - 2h\overline{V}^j_l) \tag{21}$$

Expanding each term on the right-hand side of (21) in Taylor series about (x_l, t_j) and equating the coefficients up to h^4 to zero, we obtain $(c_0, c_1, c_2) = (\frac{1}{2}, \frac{1}{4}, \frac{3}{2})$. Thus, we have

$$\begin{aligned}\overline{\overline{U}}^j_{xx_{l+1/2}} &= \frac{1}{2h^2}(\overline{U}^j_{l+1} - 2\overline{U}^j_l + \overline{U}^j_{l-1}) + \frac{1}{4h}(\overline{V}^j_{l+1} - \overline{V}^j_{l-1}) + \frac{3}{2h^2}(\overline{U}^j_{l+1} - \overline{U}^j_{l-1} - 2h\overline{V}^j_l) \\ &= U^j_{xx_{l+1/2}} + \theta k U_{21} + O(k^2 + hk + h^3)\end{aligned} \tag{22}$$

Similarly,

$$\begin{aligned}\overline{\overline{U}}^j_{xx_{l-1/2}} &= \frac{1}{2h^2}(\overline{U}^j_{l+1} - 2\overline{U}^j_l + \overline{U}^j_{l-1}) + \frac{1}{4h}(\overline{V}^j_{l+1} - \overline{V}^j_{l-1}) - \frac{3}{2h^2}(\overline{U}^j_{l+1} - \overline{U}^j_{l-1} - 2h\overline{V}^j_l) \\ &= U^j_{xx_{l-1/2}} + \theta k U_{21} + O(k^2 + hk + h^3)\end{aligned} \tag{23}$$

Further, suppose

$$\begin{aligned}h^3\overline{\overline{V}}^j_{xx_{l+1/2}} =& d_0(\overline{U}^j_{l+1} - \overline{U}^j_{l-1} - 2h\overline{V}^j_l) + d_1 h(\overline{V}^j_{l+1} + \overline{V}^j_{l-1} - 2\overline{V}^j_l) + d_2(\overline{U}^j_{l+1} + \overline{U}^j_{l-1} - 2\overline{U}^j_l) \\ &+ d_3 h(\overline{V}^j_{l+1} - \overline{V}^j_{l-1})\end{aligned} \tag{24}$$

Expanding each term on the right-hand side of (24) in Taylor series about (x_l, t_j) and equating the coefficients up to h^5 to zero, we obtain $(d_0, d_1, d_2, d_3) = (-\frac{15}{4}, \frac{9}{4}, -6, 3)$. Hence the approximation

$$\begin{aligned}\overline{\overline{V}}^j_{xx_{l+1/2}} =& -\frac{15}{4h^3}(\overline{U}^j_{l+1} - \overline{U}^j_{l-1} - 2h\overline{V}^j_l) + \frac{9}{4h^2}(\overline{V}^j_{l+1} + \overline{V}^j_{l-1} - 2\overline{V}^j_l) - \frac{6}{h^3}(\overline{U}^j_{l+1} + \overline{U}^j_{l-1} - 2\overline{U}^j_l) \\ &+ \frac{3}{h^2}(\overline{V}^j_{l+1} - \overline{V}^j_{l-1})\end{aligned}$$

$$= V^{j}_{xx_{l+1/2}} + \theta k U_{31} + O(k^2 + hk + h^3) \tag{25}$$

Likewise,

$$\begin{aligned}\overline{\overline{V}}^{j}_{xx_{l-1/2}} &= -\frac{15}{4h^3}(\overline{U}^{j}_{l+1} - \overline{U}^{j}_{l-1} - 2h\overline{V}^{j}_{l}) + \frac{9}{4h^2}(\overline{V}^{j}_{l+1} + \overline{V}^{j}_{l-1} - 2\overline{V}^{j}_{l}) + \frac{6}{h^3}(\overline{U}^{j}_{l+1} + \overline{U}^{j}_{l-1} - 2\overline{U}^{j}_{l}) \\ &\quad - \frac{3}{h^2}(\overline{V}^{j}_{l+1} - \overline{V}^{j}_{l-1}), \\ &= V^{j}_{xx_{l-1/2}} + \theta k U_{31} + O(k^2 + hk + h^3)\end{aligned} \tag{26}$$

Using the approximations (20.1)–(20.4) and (21)–(26), we consider

$$\overline{\overline{F}}^{j}_{l\pm1/2} = f(x_{l\pm1/2}, \bar{t}_j, \overline{\overline{U}}^{j}_{l\pm1/2}, \overline{\overline{V}}^{j}_{l\pm1/2}, \overline{\overline{U}}^{j}_{xx_{l\pm1/2}}, \overline{\overline{V}}^{j}_{xx_{l\pm1/2}}) = F^{j}_{l\pm1/2} + \theta k T_0 + O(kh + h^3 + k^2) \tag{27}$$

Then, the difference formula of $O(k^2 + kh^2 + h^4)$ for the equation (1) and the corresponding difference method for u_x may be written in coupled manner as:

$$\begin{aligned}&\left[\overline{A}^{j}_{l} - \frac{2h^2}{15}\left(\frac{\overline{A}^{j}_{x_l}}{\overline{A}^{j}_{l}}\right) + \frac{h^2}{15}\overline{A}^{j}_{xx_l}\right]\left[-2\left(\overline{U}^{j}_{l+1} - 2\overline{U}^{j}_{l} + \overline{U}^{j}_{l-1}\right) + h\left(\overline{V}^{j}_{l+1} - \overline{V}^{j}_{l-1}\right)\right] \\ &+ \frac{h^4}{90}\left[\left(1 - \frac{h\overline{A}^{j}_{x_l}}{\overline{A}^{j}_{l}}\right)\overline{U}^{j}_{t_{l+1}} + 13\overline{U}^{j}_{t_l} + \left(1 + \frac{h\overline{A}^{j}_{x_l}}{\overline{A}^{j}_{l}}\right)\overline{U}^{j}_{t_{l-1}}\right] \\ &= \frac{h^4}{90}\left[\left(4 - \frac{2h\overline{A}^{j}_{x_l}}{\overline{A}^{j}_{l}}\right)\overline{\overline{F}}^{j}_{l+1/2} + 7\overline{\overline{F}}^{j}_{l} + \left(4 + \frac{2h\overline{A}^{j}_{x_l}}{\overline{A}^{j}_{l}}\right)\overline{\overline{F}}^{j}_{l-1/2}\right] + \overline{T}^{j^{(1)}}_{l},\end{aligned} \tag{28.1}$$

$$\begin{aligned}&\overline{A}^{j}_{l}\left[-3\left(\overline{U}^{j}_{l+1} - \overline{U}^{j}_{l-1}\right) + h\left(\overline{V}^{j}_{l+1} + 4\overline{V}^{j}_{l} + \overline{V}^{j}_{l-1}\right)\right] + \frac{h^4}{60}\left[\overline{U}^{j}_{t_{l+1}} - \overline{U}^{j}_{t_{l-1}} - 2h\left(\frac{\overline{A}^{j}_{x_l}}{\overline{A}^{j}_{l}}\right)\overline{U}^{j}_{t_l}\right] \\ &= \frac{h^4}{30}\left[\overline{F}^{j}_{l+1/2} - \overline{F}^{j}_{l-1/2} - h\left(\frac{\overline{A}^{j}_{x_l}}{\overline{A}^{j}_{l}}\right)\overline{F}^{j}_{l}\right] + \overline{T}^{j^{(2)}}_{l}, \quad l = 1, \ldots, N, \quad j = 0, 1, \ldots,\end{aligned} \tag{28.2}$$

respectively, where $\overline{T}^{j^{(1)}}_{l} = O(k^2h^4 + kh^6 + h^8)$, $\overline{T}^{j^{(2)}}_{l} = O(k^2h^3 + kh^5 + h^7)$, for $\theta = 1/2$.

Using the approximations (19), (27) and differentiating equation (7) w.r.t. 't', we obtain

$$\begin{aligned}&\frac{h^4}{90}\left[\left(4 - \frac{2h\overline{A}^{j}_{x_l}}{\overline{A}^{j}_{l}}\right)\overline{\overline{F}}^{j}_{l+1/2} + 7\overline{\overline{F}}^{j}_{l} + \left(4 + \frac{2h\overline{A}^{j}_{x_l}}{\overline{A}^{j}_{l}}\right)\overline{\overline{F}}^{j}_{l-1/2}\right] \\ &= \frac{h^4}{90}\left[\left(4 - \frac{2hA^{j}_{x_l}}{A^{j}_{l}}\right)F^{j}_{l+1/2} + 7F^{j}_{l} + \left(4 + \frac{2hA^{j}_{x_l}}{A^{j}_{l}}\right)F^{j}_{l-1/2}\right] + \frac{\theta kh^4}{6}(A_{00}U_{41} + A_{01}U_{40} + U_{02}) \\ &+ O(k^2h^4 + kh^6 + h^8)\end{aligned} \tag{29}$$

With the help of relations (8), (29) and the approximations (9.1)–(9.15), the local truncation error $\overline{T}^{j^{(1)}}_{l}$ associated with the difference formula (28.1) is obtained as

$$\overline{T}^{j^{(1)}}_{l} = -\frac{kh^4}{6}\left(\theta - \frac{1}{2}\right)U_{02} + O(k^2h^4 + kh^6 + h^8) \tag{30}$$

For the proposed difference scheme (28.1) to be of $O(k^2 + kh^2 + h^4)$, the coefficient of kh^4 in (30) must be zero. Consequently, we obtain $\theta = 1/2$ and the local truncation error associated with (28.1) becomes $\overline{T}^{j^{(1)}}_{l} = O(k^2h^4 + kh^6 + h^8)$.

Similarly, it is seen that $\overline{T}^{j^{(2)}}_{l} = O(k^2h^3 + kh^5 + h^7)$ and thus the difference scheme (28.2) is an $O(k^2 + kh^2 + h^4)$ approximation for the derivative $\frac{\partial u}{\partial x} = v$.

Incorporating the initial and boundary conditions, the three-point compact difference scheme (28.1)–(28.2) has the block tridiagonal matrix structure, to which one can apply suitable iterative methods. If the differential equation is linear, we can solve the linear system using the block Gauss–Seidel iteration method; in the nonlinear case, we can use the Newton's nonlinear block iteration method (see Kelly [17], Hageman and Young [14]).

3. Stability analysis

Let us consider a class of linear singular equation of the form:

$$\nabla^4 u + \frac{\partial u}{\partial t} \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{\alpha}{r}\frac{\partial}{\partial r}\right)^2 u + \frac{\partial u}{\partial t} = f(r,t), \ \ 0 < r < 1, \ t > 0 \tag{31}$$

equipped with the initial condition (2) and boundary conditions of the form (3.1)–(3.2). Equivalently, equation (31) can be written as:

$$\frac{\partial^4 u}{\partial r^4} + \frac{\partial u}{\partial t} = B(r)\frac{\partial^3 u}{\partial r^3} + C(r)\frac{\partial^2 u}{\partial r^2} + D(r)\frac{\partial u}{\partial r} + f(r,t), \ 0 < r < 1, \ t > 0, \tag{32}$$

where

$$B(r) = \frac{-2\alpha}{r}, \ \ C(r) = \frac{\alpha(2-\alpha)}{r^2}, \ \ D(r) = \frac{\alpha(\alpha-2)}{r^3}.$$

For $\alpha = 1$ and 2,

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{\alpha}{r}\frac{\partial}{\partial r}$$

denotes the Laplacian operator in cylindrical and spherical coordinates, respectively, in one space dimension. It has been experienced in the past that the high accuracy numerical solution usually deteriorates in the vicinity of the singularity $r = 0$, whereas the proposed difference method (28.1)–(28.2) is applicable to 1D fourth order parabolic equations irrespective of coordinates, that is, the proposed method is directly applicable to solve equation (32). We do not require any modification in the derived scheme unlike the method discussed in [28]. Thus the numerical scheme for problems in polar coordinates is of significance in this discussion.

Replacing the variable x by r and applying the difference method (28.1)–(28.2) to the singular equation (32), we obtain:

$$\begin{aligned}
&\left[-2\left(\overline{u}_{l+1}^j - 2\overline{u}_l^j + \overline{u}_{l-1}^j\right) + h\left(\overline{v}_{l+1}^j - \overline{v}_{l-1}^j\right)\right] + \frac{h^4}{90}\left[\overline{u}_{t_{l+1}}^j + 13\overline{u}_{t_l}^j + \overline{u}_{t_{l-1}}^j\right] \\
&= \frac{h^4}{90}\Big[4\left(B_{l+1/2}\overline{\overline{v}}_{rr_{l+1/2}}^j + C_{l+1/2}\overline{\overline{u}}_{rr_{l+1/2}}^j + D_{l+1/2}\overline{\overline{v}}_{l+1/2}^j + \overline{f}_{l+1/2}^j\right) \\
&+ 7\left(B_l\overline{\overline{v}}_{rr_l}^j + C_l\overline{\overline{u}}_{rr_l}^j + D_l\overline{\overline{v}}_l^j + \overline{f}_l^j\right) + 4\left(B_{l-1/2}\overline{\overline{v}}_{rr_{l-1/2}}^j + C_{l-1/2}\overline{\overline{u}}_{rr_{l-1/2}}^j + D_{l-1/2}\overline{\overline{v}}_{l-1/2}^j + \overline{f}_{l-1/2}^j\right)\Big],
\end{aligned} \tag{33.1}$$

$$\begin{aligned}
&\left[-3\left(\overline{u}_{l+1}^j - \overline{u}_{l-1}^j\right) + h\left(\overline{v}_{l+1}^j + 4\overline{v}_l^j + \overline{v}_{l-1}^j\right)\right] + \frac{h^4}{60}\left[\overline{u}_{t_{l+1}}^j - \overline{u}_{t_{l-1}}^j\right] \\
&= \frac{h^4}{30}\Big[B_{l+1/2}\overline{\overline{v}}_{rr_{l+1/2}}^j + C_{l+1/2}\overline{\overline{u}}_{rr_{l+1/2}}^j + D_{l+1/2}\overline{\overline{v}}_{l+1/2}^j + \overline{f}_{l+1/2}^j - B_{l-1/2}\overline{\overline{v}}_{rr_{l-1/2}}^j - C_{l-1/2}\overline{\overline{u}}_{rr_{l-1/2}}^j \\
&- D_{l-1/2}\overline{\overline{v}}_{l-1/2}^j - \overline{f}_{l-1/2}^j\Big]
\end{aligned} \tag{33.2}$$

where $\overline{\overline{v}}_{l\pm1/2}^j$, $\overline{\overline{u}}_{rr_{l\pm1/2}}^j$, $\overline{\overline{v}}_{rr_{l\pm1/2}}^j$, $\overline{\overline{v}}_l^j$, $\overline{\overline{u}}_{rr_l}^j$, $\overline{\overline{v}}_{rr_l}^j$ are defined in Section 2 and for $p = 0, \pm 1/2$: $B_{l+p} = B(r_{l+p})$, $C_{l+p} = C(r_{l+p})$, $D_{l+p} = D(r_{l+p})$, $\overline{f}_{l+p}^j = f(r_{l+p}, \overline{t}_j)$.

Note that, the difference scheme (33.1)–(33.2) is of $O(k^2 + kh^2 + h^4)$ for the solution of differential equation (31) and is free from the terms $1/r_{l\pm1}$, hence, can be very easily solved for $l = 1, \ldots, N$ in the region $[0 < r < 1] \times [t > 0]$ without any modification unlike the three-point compact difference scheme of $O(k^2 + h^4)$ proposed in [28] which contain the term f_{l-1}, so a singularity arises at $l = 1$ if $r_0 = 0$ and requires a special treatment to deal with the singular points. However, this is not the case with our proposed scheme as $f_{l-1/2}$ appears instead of f_{l-1}, which this is the major advantage of using half-step discretization.

Next, we discuss the stability of the derived scheme (28.1)–(28.2) using von Neumann analysis for the following linear model problem:

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial u}{\partial t} = f(x,t), \ \ \ 0 < x < 1, \ \ t > 0 \tag{34}$$

The proposed difference method (28.1)–(28.2) applied to equation (34) can be written as:

$$(u_{l-1}^{j+1} + 13u_l^{j+1} + u_{l+1}^{j+1}) - 90r(u_{l-1}^{j+1} - 2u_l^{j+1} + u_{l+1}^{j+1}) + 45rh(v_{l+1}^{j+1} - v_{l-1}^{j+1})$$

$$= (u_{l-1}^j + 13u_l^j + u_{l+1}^j) + 90r(u_{l-1}^j - 2u_l^j + u_{l+1}^j) - 45rh(v_{l+1}^j - v_{l-1}^j) + \Sigma f \tag{35.1}$$

$$(1-90r)(u_{l+1}^{j+1} - u_{l-1}^{j+1}) + 30rh(v_{l-1}^{j+1} + 4v_l^{j+1} + v_{l+1}^{j+1})$$
$$= (1+90r)(u_{l+1}^j - u_{l-1}^j) - 30rh(v_{l-1}^j + 4v_l^j + v_{l+1}^j) + \Sigma f_1,\ \ l = 1, \dots, N,\ \ j = 0, 1, \dots \tag{35.2}$$

where $r = k/h^4$, $\Sigma f = k[4\overline{f}_{l-1/2}^j + 7\overline{f}_l^j + 4\overline{f}_{l+1/2}^j]$ and $\Sigma f_1 = 2k[\overline{f}_{l+1/2}^j - \overline{f}_{l-1/2}^j]$.

To apply the Von Neumann method, we assume that the solutions are given by

$$u_l^j = \eta^j e^{i\beta l},$$
$$v_l^j = \gamma^j e^{i\beta l},$$

for $i = \sqrt{-1}$, η, γ being the amplification factors and β being the phase angle. Using these in the homogenous part of the scheme (35.1)–(35.2) gives the following matrix vector form:

$$\begin{bmatrix} \eta \\ \gamma \end{bmatrix}^{j+1} = \boldsymbol{M} \begin{bmatrix} \eta \\ \gamma \end{bmatrix}^{j}, \tag{36}$$

where the amplification matrix $\boldsymbol{M}$ is given by

$$\boldsymbol{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} p & -b \\ q & -d \end{bmatrix}$$

and

$$a = 15 - 4\sin^2\frac{\beta}{2} + 360r\sin^2\frac{\beta}{2}, \quad b = 180rh\sin\frac{\beta}{2}\cos\frac{\beta}{2}i,$$
$$c = 4i(1-90r)\sin\frac{\beta}{2}\cos\frac{\beta}{2}, \quad d = 60rh(3 - 2\sin^2\frac{\beta}{2}),$$
$$p = 15 - 4\sin^2\frac{\beta}{2} - 360r\sin^2\frac{\beta}{2}, \quad q = 4i(1+90r)\sin\frac{\beta}{2}\cos\frac{\beta}{2}.$$

The von Neumann necessary condition for the stability of the system (36) is that the maximum modulus of the eigenvalues of the amplification matrix $\boldsymbol{M}$ are to be less than or equal to one. By direct calculation, the eigenvalues of the matrix $\boldsymbol{M}$ are given by

$$\nu_1 = -1, \quad \nu_2 = \frac{dp - bq}{ad - bc}$$

Simplifying, we get

$$\nu_2 = \frac{45 - 30\sin^2\frac{\beta}{2} - 4\sin^4\frac{\beta}{2} - 360r\sin^4\frac{\beta}{2}}{45 - 30\sin^2\frac{\beta}{2} - 4\sin^4\frac{\beta}{2} + 360r\sin^4\frac{\beta}{2}}$$

Since $\min(45 - 30\sin^2\frac{\beta}{2} - 4\sin^4\frac{\beta}{2}) = 11 > 0$, for all phase angles β, hence the derived difference scheme applied to the linear equation (34) is unconditionally stable.

4. Numerical illustrations

In this section, we implement the proposed difference method over a collection of problems having physical relevance. The exact solution of the problems are provided in each case. The initial and boundary conditions correspond to the data from the exact solution. The proposed difference method can be expressed in block tridiagonal matrix form. We have used iterative methods for solving the coupled system of equations at each mesh point. If the differential equation is linear, then the resulting system is solved by block Gauss–Seidel iteration method whereas the nonlinear systems are solved using Newton's nonlinear block Gauss–Seidel iteration method (see Kelly [17], Saad [30] and Hageman and Young [14]) and in each case the iterations are terminated once the absolute error tolerance $\le 10^{-10}$ is achieved.

Example 1. Consider the GKS equation (4) with $\alpha = \gamma = 1$ and $\beta = 4$ on the interval $[-1, 1]$. The exact solution is given by

$$u(x,t) = 9 + 2c + 15\tanh\theta - 15\tanh^2\theta - 15\tanh^3\theta$$

with $\theta = -\frac{1}{2}x + ct$. The L_∞ and L_2 errors obtained by the presented method for $c = 1$ are listed in Table 1 at time $t = 1$ for different values of h and k and are compared with the results of [22]. Also, in Table 2, the L_∞ errors for different values of c at time $t = 1$ and $t = 10$ for $k = 0.05$, $h = 1/32$ and the used CPU time (in seconds) are reported. The obtained results are compared with the methods proposed in [18,22]. The two dimensional graph of numerical solution *vs* exact solution is plotted in Fig. 1 at different time levels for $-1 < x < 1$.

Table 1
Comparison of L_∞ and L_2 errors for $c = 1$ at $t = 1$ for different values of h and k, Example 1.

k	Proposed method $h = 1/16$ L_∞	Method [22] $(J = 4)$ L_∞	Proposed method $h = 1/16$ L_2	Method [22] $(J = 4)$ L_2	Proposed method $h = 1/32$ L_∞	Method [22] $(J = 5)$ L_∞	Proposed method $h = 1/32$ L_2	Method [22] $(J = 5)$ L_2
0.1	1.0080e−003	7.3e−003	5.2497e−004	6.5e−003	1.9476e−004	4.4e−003	1.2490e−004	3.7e−003
0.01	1.2003e−005	6.1e−003	6.5234e−006	5.7e−003	1.9417e−006	3.3e−005	1.2532e−006	2.6e−005
0.001	8.1227e−008	1.0e−004	4.6888e−008	8.4e−005	7.1686e−011	2.8e−006	2.9184e−011	1.5e−006

Table 2
Comparison of L_∞ errors for $h = 1/32$ and $k = 0.05$ at $t = 1$ for different values of c with [18,22], Example 1.

t	c	Proposed method	CPU time(s)	Method [22]	Method [18]
1	0.1	1.3086e−006	95.7471	7.7e−007	2.6e−004
	0.01	3.1832e−012	0.0738	1.8e−006	3.2e−005
	0.001	1.1067e−012	0.0841	1.6e−006	3.2e−005
10	0.1	5.3465e−007	145.0661	–	–
	0.01	5.8886e−012	0.8045	–	–
	0.001	2.8813e−012	0.5361	–	–

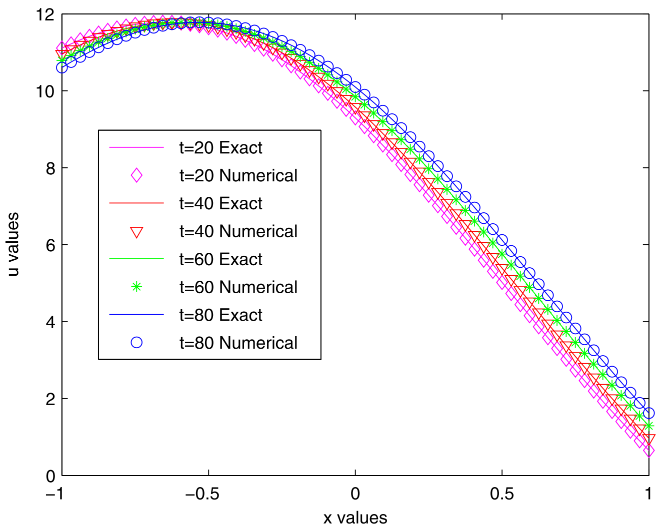


Fig. 1. Example 1: Comparison between numerical and exact solution at different times.

Example 2. In this example, we consider the KS equation (5) for $\alpha = 1$ and $\gamma = 0.5$ on the interval $[-1, 1]$. The exact solution is given by

$$u(x,t) = -\frac{0.1}{K} + \frac{60}{19}K(-38\gamma K^2 + \alpha)\tanh\theta + 120\gamma K^3 \tanh^3\theta$$

with $\theta = Kx + 0.1t$ and $K = (1/2)\sqrt{11\alpha/19\gamma}$. The L_∞, L_2 errors and the CPU time using the derived difference method are listed in Table 3 at time $t = 1$ and $t = 5$ for $h = 1/16$ for different values of k and compared with the results in [22,29]. We show the two dimensional visual comparison of exact and numerical solution at $t = 0.5$ with $h = 1/16$ and $k = 0.1$ in Fig. 2.

Example 3. Consider the GKS equation (4) for $\alpha = \gamma = 1$ and $\beta = \frac{12}{\sqrt{47}}$. The exact solution is given by

$$u(x,t) = -\frac{396.8}{47\sqrt{47}} + \frac{15}{47\sqrt{47}}\left[3\tanh\theta - 3\tanh^2\theta + \tanh^3\theta\right]$$

with $\theta = (1/2\sqrt{47})x + 0.1t$. The computational domain is taken as $[-1, 1]$. The L_∞ errors and the used CPU time by the proposed difference method with $h = 1/8$ and the same time step $k = 0.05$ as considered in [29] at various time levels

Table 3
Comparison of L_∞ and L_2 errors at $t = 1$ with $h = 1/16$ for different values k, Example 2.

t	k	Proposed method L_2	Proposed method L_∞	CPU time(s)	Method [29] L_2	Method [29] L_∞	Method [22] L_2	Method [22] L_∞
1	0.1	8.4966e−006	1.5541e−005	8.1471	1.4e−004	1.3e−004	9.9e−003	8.9e−003
	0.01	8.4715e−007	1.2716e−006	9.4960	1.5e−006	1.3e−006	2.1e−003	1.6e−003
5	0.1	4.9036e−007	1.6258e−006	20.8869	–	–	–	–
	0.01	1.0305e−007	1.6337e−007	44.3857	–	–	–	–

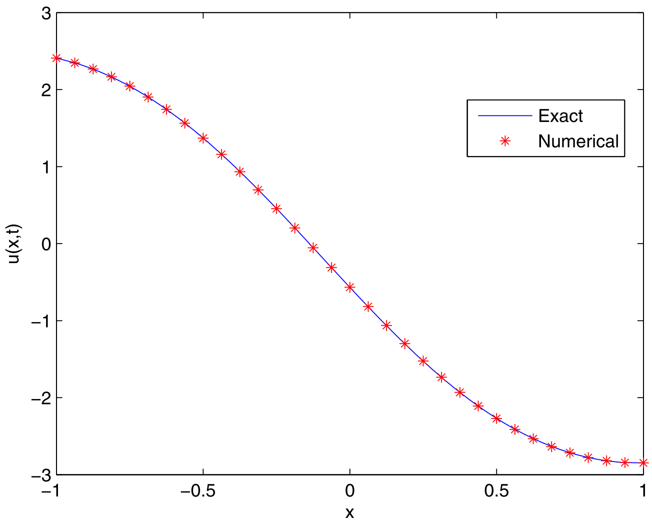


Fig. 2. Plot of exact-numerical solution at $t = 0.5$ with $k = 0.1$ and $h = 1/16$, Example 2.

Table 4
Comparison of L_∞ errors for $h = 1/8$ and $k = 0.05$ at different times, Example 3.

t	Proposed method	CPU time(s)	Method [29]	Method [18]
0.2	1.3789e−012	0.0327	2.41e−010	2.38e−007
0.4	1.0041e−012	0.0354	2.38e−010	2.38e−007
0.6	1.1735e−012	0.0386	6.36e−010	2.38e−007
0.8	1.4309e−012	0.0397	1.05e−009	1.19e−007
1.0	1.6698e−012	0.0413	5.95e−010	2.38e−007
3.0	5.7070e−012	0.0618	3.01e−009	–
5.0	1.6240e−011	0.0914	3.05e−009	–
10	1.0315e−011	0.1675	6.88e−010	–
15	1.1280e−011	0.5323	–	–
20	2.8906e−011	0.6261	–	–

are reported in Table 4 and compared with the results obtained in [18,29]. It is observed from Table 4 that the proposed method is more accurate than the methods proposed in [18,29]. Fig. 3 exhibits the exact and numerical solutions at $t = 1$ for $h = 1/16$ and $k = 0.05$.

Example 4. In this example, we solve the KS equation (5) for $\alpha = 1$ and $\gamma = 1$ which represents the simplest nonlinear PDE exhibiting chaotic behavior over a finite spatial domain with the Gaussian initial condition [22]

$$u(x, 0) = \exp(-x^2) \tag{37}$$

and the boundary conditions

$$u(a, t) = 0, \;\; u(b, t) = 0, \;\; \frac{\partial u}{\partial x}(a, t) = 0, \;\; \frac{\partial u}{\partial x}(b, t) = 0 \tag{38}$$

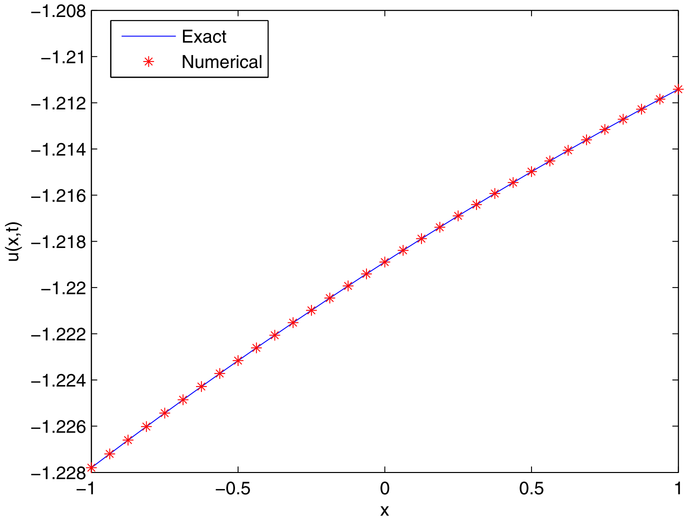
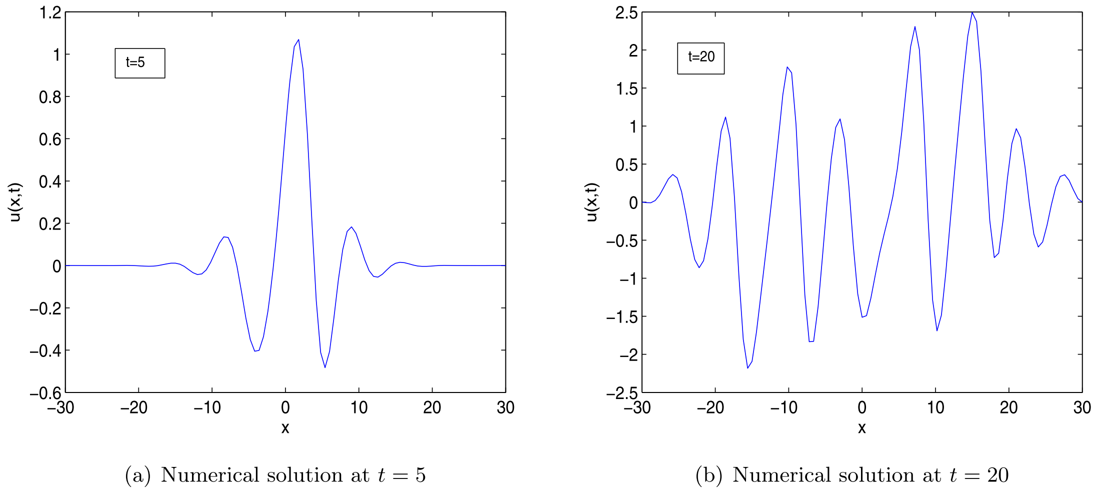


Fig. 3. Plot of exact-numerical solution at $t = 1$ with $k = 0.05$ and $h = 1/16$, Example 3.



(a) Numerical solution at $t = 5$ (b) Numerical solution at $t = 20$

Fig. 4. The chaotic solution of the KS equation (5) for $\alpha = 1$ and $\gamma = 1$ with the Gaussian initial condition (37) at $t = 5$ and $t = 20$, Example 4.

The computational domain is taken as $[a, b] = [-30, 30]$ with $k = 0.01$ and number of partitions as 100. The numerical solution at $t = 5$ and $t = 20$ is presented in Fig. 4(a) and Fig. 4(b), respectively. It is observed that the results present same characteristics as in [27] and are numerically convergent for the chaotic behavior too.

Example 5. We solve numerically the linear singular problem (31) whose exact solution is $u = r^4 \sin r \sin t$ using difference scheme (33.1)–(33.2). The L_2 errors in u and u_r are reported in Table 5 at $t = 1$ for $\alpha = 1, 2$ and compared with the results given in [28]. We see that our obtained results are better in comparison with those given in [28].

Example 6. Consider the following non-homogenous EFK equation

$$\frac{\partial u}{\partial t} + \gamma \frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u}{\partial x^2} + f(u) = g(x,t), \quad 0 < x < 1, \ \ t > 0 \tag{39}$$

where $f(u) = u^3 - u$. Here

$$g(x,t) = \left(\gamma \pi^4 + \pi^2 + e^{-2t} \sin^2(\pi x) - 2\right) e^{-t} \sin(\pi x).$$

The exact solution of the above equation is $u(x,t) = e^{-t} \sin(\pi x)$. The L_∞ errors and order of convergence for the proposed method are reported in Table 6 at $t = 1$ for various values of γ for $h = \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$ with $\lambda = 1.6$. For a fixed parameter value

Table 5
Comparison of L_2 errors for Example 5 at $t = 1$.

h		Proposed $O(k^2 + kh^2 + h^4)$-method (33.1)–(33.2)		$O(k^2 + kh^2 + h^4)$-method discussed in [28]	
		$\alpha = 1$	$\alpha = 2$	$\alpha = 1$	$\alpha = 2$
1/8	u	2.2170(−05)	5.8983(−05)	1.395(−04)	8.408(−05)
	u_r	1.0755(−04)	1.5511(−04)	5.270(−04)	4.053(−04)
1/16	u	1.3370(−06)	4.5719(−06)	8.902(−06)	1.999(−06)
	u_r	7.2343(−06)	1.5201(−05)	3.769(−05)	1.532(−05)
1/32	u	2.8064(−08)	1.3956(−07)	4.699(−07)	1.309(−07)
	u_r	3.0896(−07)	7.1765(−07)	2.151(−06)	5.920(−07)

Table 6
L_∞ errors for different values of γ at $t = 1$ with $\lambda = (k/h^2) = 1.6$, Example 6.

h		L_∞ $\gamma = 0.1$	Order $\gamma = 0.1$	L_∞ $\gamma = 0.01$	Order $\gamma = 0.01$	L_∞ $\gamma = 0.001$	Order $\gamma = 0.001$
1/8	u	6.9714e–006	–	8.8221e–006	–	8.2477e–006	–
	u_x	2.4196e–005	–	5.3812e–005	–	2.9893e–004	–
1/16	u	4.2780e–007	4.0264	5.2839e–007	4.0614	5.6493e–007	3.8678
	u_x	1.3437e–006	4.1705	2.0352e–006	4.7247	1.8846e–005	3.9875
1/32	u	2.6111e–008	4.0342	3.2680e–008	4.0151	3.6523e–008	3.9512
	u_x	8.1111e–008	4.0502	1.0890e–007	4.2241	1.3317e–006	3.8229

Table 7
L_∞ errors at $t = 1$ and $t = 2$ with $\lambda = (k/h^2) = 1.6$, Example 7.

h		L_∞	Order	L_∞	Order
		$t = 2$		$t = 1$	
1/8	u	7.3353(−04)	–	5.7259(−04)	–
	u_x	2.8520(−02)	–	1.5785(−02)	–
1/16	u	4.7348(−05)	3.9535	3.6933(−05)	3.9545
	u_x	1.8378(−03)	3.9559	1.0146(−03)	3.9596
1/32	u	2.9823(−06)	3.9888	2.3246(−06)	3.9899
	u_x	1.1564(−04)	3.9903	6.3815(−05)	3.9909

λ, the proposed method (28.1)–(28.2) behaves like a fourth order method which the computed results in Table 6 exhibit for various values of γ.

Example 7. We consider the one-dimensional fourth order parabolic PDE with variable coefficient of the form:

$$(1 + x^2 + t^2)\frac{\partial^4 u}{\partial x^4} + \frac{\partial u}{\partial t} = f(x,t), \quad 0 < x < 1, \ \ t > 0 \tag{40}$$

where $f(x,t) = \left[\pi^4\left(1 + x^2 + t^2\right) - 1\right]e^{-t}\sin\pi x$ and the exact solution is $u(x,t) = e^{-t}\sin\pi x$. The maximum absolute errors and order of convergence for $u(x,t)$ and $u_x(x,t)$ using the difference scheme (28.1)–(28.2) are tabulated in Table 7 at $t = 1$ and $t = 2$ for $\lambda = 1.6$ with $N + 1 = 8, 16$ and 32. It is exhibited from Table 7 that the proposed discretization produces fourth order accurate results for a fixed value of the mesh ratio parameter λ, i.e. when $k \propto h^2$.

5. Concluding remarks

In essence, the current work presents highly accurate numerical approximation of order two in time and four in space for the 1D unsteady biharmonic equation subject to Dirichlet and Neumann boundary conditions using half-step discretization. Mohanty [28] has already developed fourth order accurate in space three-point compact scheme for the solution of 1D unsteady biharmonic problem of first kind but that method is not directly applicable to problems in polar coordinates and requires a special technique to handle singular points because of the presence of the terms of the form $1/r_{l-1}$ which give rise to singularity at $l = 1$ if $r_0 = 0$. In the present paper, by using half-step nodal points, the singularity at $r = 0$ is avoided which enables the direct application of the proposed method for fourth order parabolic equations with singular coefficients. The method is derived using three spatial grid points, hence, in each time step, every iteration involves solving a tridiagonal system. The proposed two-level implicit method is shown to be unconditionally stable for a model linear problem. The value of the derivative u_x is already computed and available at all grid points and need not be further approximated from the computed value of the solution u. Extensive numerical results of diverse scenarios for the KS equation illustrates the superiority of our approach. It is observed that the simulating results are in good agreement with both the exact and existing numerical solutions [18,22,29]. The solutions found in this work will be useful in studying electromagnetic waves, fluid flows, reaction–diffusion–combustion dynamics, stability of the output in electricity current, magneto hydrodynamics, turbulence in microtides and other physical processes. The proposed method will enable the development of stable algorithms for solving other nonlinear unsteady biharmonic problems in applied sciences and engineering. However, the current approach is not applicable for the boundary conditions of second kind, i.e., when u and $\partial^2 u/\partial n^2$ are prescribed at the boundary. The two-level method proposed in this work does not work if the second order time derivative term is present in the differential equation, for which a modified three-level method is required. The method derived in the present study is applicable for 1D time dependent biharmonic problems only. An investigation to extend the current approach to solve 2D and 3D unsteady biharmonic problems is underway.

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