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# High accuracy implicit variable mesh methods for numerical study of special types of fourth order non-linear parabolic equations

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ABSTRACT

In this paper, we propose a new implicit high order two-level finite difference method of $O(k^2 + kh_l + h_l^3)$ for the solution of special type of 1D fourth-order non-linear parabolic partial differential equation of the form $u_{xxxx} - 2u_{xxt} + u_{tt} = f(x, t, u, u_x, u_{xx} - u_t, u_{xxx} - u_{xt})$, $a < x < b$, $t > 0$ and a new three-level finite difference method of $O(k^2 + h_l^3)$ for the solution of another type of 1D fourth-order non-linear parabolic partial differential equation of the form $A(x, t, u, u_{xx})u_{xxxx} + u_{tt} = g(x, t, u, u_t, u_x, u_{xx}, u_{xxx})$, $a < x < b$, $t > 0$, both on a variable mesh, subject to suitable initial and Dirichlet boundary conditions, where $k > 0$ and $h_l > 0$ are the mesh sizes in time and space coordinates, respectively. The third order variable mesh methods proposed are based on only three spatial grid points $x_l, x_{l\pm1}$, meaning that no fictitious points are required for incorporating the boundary conditions. We discuss how our formulation is able to handle linear singular problem. The proposed two-level finite difference method is shown to be unconditionally stable for a model linear problem. We applied the proposed methods to find the numerical solution of more complex fourth-order non-linear equations like second-order Benjamin–Ono equation and the good Boussinesq equation. It is evident from the numerical experiments that the numerical results agree well with the exact solutions, hence demonstrating the efficiency of the methods derived.



## 1. Introduction

Fourth-order partial differential equations (PDEs) model various physical phenomenon and dynamic processes in physics, engineering and geological sciences including transverse vibrations of a uniform flexible beam, deformation of beams, viscoelastic and inelastic flows, plate deflection theory (see [1–3]). These equations arise of necessity, subsequently they are studied in these various scientific contexts. Several techniques for finding the analytical solution of non-linear fourth-order PDEs have been discussed, Adomian Decomposition Method (ADM) [4,5] and variation iteration method (VIM) [6]. In 2009, Dehghan and Manafian [7] applied the homotopy perturbation method to solve both linear and non-linear boundary value problems for fourth-order parabolic PDEs with variable coefficients. Recently, Khan et al. [8] approached analytically the fourth-order parabolic PDE with variable coefficients by employing the homotopy analysis method. Since analytical solutions of most non-linear fourth-order

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parabolic PDEs are not available, so it is required to develop efficient numerical techniques whose solutions are of great significance to scientists and engineers. Various explicit and implicit finite difference schemes of second and fourth order for the fourth-order linear parabolic equation have been studied by numerous researchers since 1950. A number of high order finite difference schemes for numerical study of transverse vibrations have been proposed by various authors [9–11], however their methods are not applicable to differential equations with singular coefficients. Khaliq and Twizell [12] studied fourth-order parabolic equations of variable coefficients via method of lines (MOL) approach to obtain a numerical approximation. In the recent past, Khan et al. [13] proposed sextic spline solution for fourth-order parabolic PDE. In [14], Caglar and Caglar have used a family of fifth degree B-spline methods for the numerical solution of fourth-order parabolic PDE. Evans and Yousif [15] have employed the alternating group explicit (AGE) method for solving fourth-order parabolic PDE. In this direction, Mohanty and Evans [16] have derived two new three-level implicit difference approximations of $O(k^2 + h^2)$ and $O(k^2 + h^4)$ in a coupled manner for 1D mildly quasi-linear fourth-order parabolic PDE using three spatial grid points on a uniform mesh. Liu et al. [17] have proposed a new numerical method called a coupling method based on a new expanded mixed finite element (EMFE) and finite element (FE) for fourth-order PDE of parabolic type by reducing the fourth-order parabolic equation to a coupled system of second-order equations and then solved a second-order equation by FE method and approximated the other one by a new EMFE method. Most recently, Siddiqi and Arshed [18] developed quintic B-spline collocation method to solve the problem of undamped transverse vibrations of a flexible straight beam. The solution of time dependent non-linear PDEs like the second-order Benjamin–Ono equation and the good Boussinesq equation is numerically challenging owing to their highly complicated mechanism of solitary wave interaction. Many finite difference numerical methods have been developed for solving such equations (see [19,20]).

Motivated and inspired by the ongoing research in this area, we present two new: two-level and three-level finite difference methods based on the approach given by Mohanty and Evans [16,21] for two different classes of non-linear fourth-order parabolic PDE on a variable mesh by reducing the fourth-order equation to a coupled system of two second-order equations using Numerov type discretization. Our methods are $O(k^2 + kh_l + h_l^3)$ and $O(k^2 + h_l^3)$ accurate, where $h_l > 0$, $l = 1(1)N$ and $k > 0$ are mesh sizes in space and time directions, respectively. In both the cases, only three spatial non-uniform grid points are needed. To the authors' knowledge no high order methods on a variable mesh for the solution of 1D non-linear fourth-order parabolic PDE have been discussed in the literature so far. For $\sigma_l = 1$ (uniform mesh case), that is, for $h_{l+1} = h_l = h$, our methods reduce to the two-level implicit finite difference method of $O(k^2 + kh^2 + h^4)$ and three-level implicit finite difference method of $O(k^2 + h^4)$, respectively. In the past, difficulties were encountered for seeking the solution of parabolic equations in polar coordinates. The solution generally deteriorates in the vicinity of singularity. To overcome this difficulty, we modified our method in a manner that the solution retains its order and accuracy everywhere even in the vicinity of singularity. The foremost advantage of developing these new methods is that these do not require any fictitious points for incorporating the boundary conditions when applied to singular problems. The proposed methods are applied to several time dependent non-linear PDEs which are significant in various fields of sciences including the complicated second-order Benjamin–Ono equation and the good Boussinesq equation.

Here is the outline of this paper. In Section 2, we present new variable mesh two-level finite difference method to solve a special type of fourth-order parabolic PDE. In Section 3, we propose new variable mesh three-level finite difference method to solve another special type of fourth-order parabolic PDE. In Section 4, the stability analysis for a linear difference equation has been discussed as a representative example and we analyze the application of new method to a linear singular problem. In Section 5, we perform numerical experiments on a large variety of problems to demonstrate the accuracy of the proposed methods. Concluding remarks are given in Section 6.

## 2. Two level implicit method

We consider the following special type of one-space dimensional fourth-order parabolic PDE of the form

$$\frac{\partial^4 u}{\partial x^4} - 2\frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial^2 u}{\partial t^2} = f(x, t, u, u_x, u_{xx} - u_t, u_{xxx} - u_{xt}), \quad a < x < b, \quad t > 0 \tag{1}$$

defined on the solution domain $\Omega \equiv \{(x, t) | a < x < b, t > 0\}$, with the initial conditions

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x), \quad a \le x \le b \tag{2}$$

and the boundary conditions are given by

$$u(a, t) = r_0(t), \quad u(b, t) = r_1(t), \quad t > 0 \tag{3.1}$$

$$u_{xx}(a, t) = s_0(t), \quad u_{xx}(b, t) = s_1(t), \quad t > 0 \tag{3.2}$$

where we assume that the functions $f, \phi, \psi, r_0, r_1, s_0, s_1$ are of sufficient smoothness and their required high order derivatives exist.

Let us denote $u_{xx} - u_t = v$, where $u$ and $v$ are functions of independent variables $x$ and $t$. Since the value of $u$ is known explicitly at $t = 0$, this implies all the successive tangential partial derivatives of $u$ are known at $t = 0$, i.e., the values of $u_x$, $u_{xx}, \ldots$ etc. are known at $t = 0$. As $v(x, 0) = u_{xx}(x, 0) - u_t(x, 0)$, this implies that the value of $v$ is known at $t = 0$. Similarly, as $u$ and $u_{xx}$ are known at $x = a$ and at $x = b$, so their tangential derivatives are known at $x = a$ and $x = b$, i.e., the values of $u_t$, $u_{tt}, \ldots, u_{xxt}, u_{xxtt}, \ldots$, etc. are known at $x = a$ and at $x = b$. Hence the values of $u$ and $v$ are known on all sides of the domain $\Omega$. So, equivalently, we reformulate problem (1)–(3) in a coupled manner as

$$\left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial t}\right)u=v,\quad (x,t)\in\Omega \tag{4.1}$$

$$\left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial t}\right)v=f(x,t,u,v,u_x,v_x),\quad (x,t)\in\Omega \tag{4.2}$$

subject to the initial conditions (2) and the boundary conditions (3.2) may be replaced by

$$v(a,t)=s_0(t)-r_0'(t),\quad v(b,t)=s_1(t)-r_1'(t),\quad t>0 \tag{5}$$

For the numerical solution of the proposed one-space dimensional parabolic problem (4.1) and (4.2), subject to the initial conditions (2) and the boundary conditions (3.1) and (5), we cover the solution domain $\Omega$ by a rectangular mesh such that $a=x_0<x_1<\ldots<x_N<x_{N+1}=b$. Let $h_l=x_l-x_{l-1}$, $l=1(1)N+1$ be the mesh sizes in $x$-direction and $k=t_{j+1}-t_j$ be the mesh spacing in $t$-direction so that $t_j=jk,\ 0\le j\le J$, where $N$ and $J$ are positive integers. The mesh ratio is denoted by $\sigma_l=(h_{l+1}/h_l)>0,\ l=1(1)N$. When $\sigma_l=1$, then it reduces to the constant mesh case. Let $U_l^j$ and $V_l^j$ denote the exact solution values of $u(x,t)$ and $v(x,t)$ at the grid point $(x_l,t_j)$ and $u_l^j$ and $v_l^j$ denote their approximate solution values, respectively.

At the grid point $(x_l,t_j)$, let

$$P_l=\sigma_l^2+\sigma_l-1,\quad Q_l=(1+\sigma_l)(1+3\sigma_l+\sigma_l^2),\quad R_l=\sigma_l(1+\sigma_l-\sigma_l^2),\quad S_l=\sigma_l(1+\sigma_l). \tag{6}$$

Let

$$\bar{t}_j=t_j+\theta k, \tag{7}$$

where $0<\theta<1$ is a parameter to be suitably determined.

In order to obtain the high accuracy method, we require the following approximations:

For $r=0,\pm1$, we denote

$$\overline{U}_{l+r}^j=\theta U_{l+r}^{j+1}+(1-\theta)U_{l+r}^j, \tag{8.1}$$

$$\overline{V}_{l+r}^j=\theta V_{l+r}^{j+1}+(1-\theta)V_{l+r}^j, \tag{8.2}$$

$$\overline{U}_{t\,l+r}^j=(U_{l+r}^{j+1}-U_{l+r}^j)/k, \tag{8.3}$$

$$\overline{V}_{t\,l+r}^j=(V_{l+r}^{j+1}-V_{l+r}^j)/k, \tag{8.4}$$

$$\overline{U}_{xl}^j=(\overline{U}_{l+1}^j-(1-\sigma_l^2)\overline{U}_l^j-\sigma_l^2\overline{U}_{l-1}^j)/(h_lS_l), \tag{8.5}$$

$$\overline{U}_{xl+1}^j=((1+2\sigma_l)\overline{U}_{l+1}^j-(1+\sigma_l)^2\overline{U}_l^j+\sigma_l^2\overline{U}_{l-1}^j)/(h_lS_l), \tag{8.6}$$

$$\overline{U}_{xl-1}^j=(-\overline{U}_{l+1}^j+(1+\sigma_l)^2\overline{U}_l^j-\sigma_l(2+\sigma_l)\overline{U}_{l-1}^j)/(h_lS_l), \tag{8.7}$$

$$\overline{V}_{xl}^j=(\overline{V}_{l+1}^j-(1-\sigma_l^2)\overline{V}_l^j-\sigma_l^2\overline{V}_{l-1}^j)/(h_lS_l), \tag{8.8}$$

$$\overline{V}_{xl+1}^j=((1+2\sigma_l)\overline{V}_{l+1}^j-(1+\sigma_l)^2\overline{V}_l^j+\sigma_l^2\overline{V}_{l-1}^j)/(h_lS_l), \tag{8.9}$$

$$\overline{V}_{xl-1}^j=(-\overline{V}_{l+1}^j+(1+\sigma_l)^2\overline{V}_l^j-\sigma_l(2+\sigma_l)\overline{V}_{l-1}^j)/(h_lS_l). \tag{8.10}$$

Eqs. (8.1), (8.2), (8.3), (8.4), (8.5)–(8.7), (8.8)–(8.10) are approximations to $u, v, u_t, v_t, u_x, v_x$, respectively at the grid point $(x_l,\bar{t}_j)$ and the neighbouring points $(x_{l\pm1},\bar{t}_j)$.

Using the above approximations, we define

$$\overline{F}_{l+r}^j=f(x_{l+r},\bar{t}_j,\overline{U}_{l+r}^j,\overline{V}_{l+r}^j,\overline{U}_{xl+r}^j,\overline{V}_{xl+r}^j). \tag{9}$$

Now, let

$$\overline{\overline{U}}_{xl}^j=\overline{U}_{xl}^j+a_lh_l[(\overline{V}_{l+1}^j-\overline{V}_{l-1}^j)+(\overline{U}_{t\,l+1}^j-\overline{U}_{t\,l-1}^j)], \tag{10.1}$$

$$\overline{\overline{V}}_{xl}^j=\overline{V}_{xl}^j+b_lh_l[(\overline{F}_{l+1}^j-\overline{F}_{l-1}^j)+(\overline{V}_{t\,l+1}^j-\overline{V}_{t\,l-1}^j)], \tag{10.2}$$

where $a_l$ and $b_l$ are the parameters to be suitably determined. Finally, let

$$\overline{\overline{F}}_l^j = f(x_l, \bar{t}_j, \overline{U}_l^j, \overline{V}_l^j, \overline{\overline{U}}_{xl}^j, \overline{\overline{V}}_{xl}^j). \tag{11}$$

Then at each internal grid point $(x_l, t_j)$, $l = 1(1)N$, $j = 0, 1, 2, \ldots$, the difference method of $O(k^2 + kh_l + h_l^3)$ for the differential Eqs. (4.1) and (4.2) are given by

$$\overline{U}_{l+1}^j - (1+\sigma_l)\overline{U}_l^j + \sigma_l\overline{U}_{l-1}^j = \frac{h_l^2}{12}[P_l(\overline{V}_{l+1}^j + \overline{U}_{t\,l+1}^j) + Q_l(\overline{V}_l^j + \overline{U}_{t\,l}^j) + R_l(\overline{V}_{l-1}^j + \overline{U}_{t\,l-1}^j)] + \overline{T}_l^{j(1)}, \tag{12.1}$$

$$\overline{V}_{l+1}^j - (1+\sigma_l)\overline{V}_l^j + \sigma_l\overline{V}_{l-1}^j = \frac{h_l^2}{12}[P_l(\overline{F}_{l+1}^j + \overline{V}_{t\,l+1}^j) + Q_l(\overline{\overline{F}}_l^j + \overline{V}_{t\,l}^j) + R_l(\overline{F}_{l-1}^j + \overline{V}_{t\,l-1}^j)] + \overline{T}_l^{j(2)}, \tag{12.2}$$

where $\overline{T}_l^{j(1)} = O(k^2h_l^2 + kh_l^3 + h_l^5)$, $\overline{T}_l^{j(2)} = O(k^2h_l^2 + kh_l^3 + h_l^5)$, provided $\sigma_l \neq 1$.

It is to be noted that the coefficients $P_l, Q_l, R_l$ in (12.1) and (12.2) are positive if $((\sqrt{5}-1)/2) < \sigma_l < ((\sqrt{5}+1)/2)$ (see Jain et al. [22]).

For the derivation of the above variable mesh difference methods (12.1) and (12.2), we follow the approaches given by Mohanty [21,23].

At the grid point $(x_l, t_j)$, for $S = A, B, C, D, U, V$ and $g$ we denote

$$S_{ab} = \frac{\partial^{a+b}S}{\partial x^a \partial t^b},$$
$$G = \frac{\partial f}{\partial t}, \quad H = \frac{\partial f}{\partial u}, \quad I = \frac{\partial f}{\partial v}, \quad J = \frac{\partial f}{\partial u_x}, \quad K = \frac{\partial f}{\partial v_x}.$$

The proposed differential Eqs. (4.1) and (4.2) at the grid point $(x_l, t_j)$ can be written as

$$U_{20} - U_{01} = V_{00} \tag{13.1}$$

$$V_{20} - V_{01} = f(x_l, t_j, U_l^j, V_l^j, U_{x\,l}^j, V_{x\,l}^j) \equiv F_l^j \ (\text{say}). \tag{13.2}$$

Similarly,

$$F_{l\pm1}^j = f(x_{l\pm1}, t_j, U_{l\pm1}^j, V_{l\pm1}^j, U_{x\,l\pm1}^j, V_{x\,l\pm1}^j). \tag{14}$$

Further, the following relations are obtained upon differentiating the system (4.1) and (4.2) with respect to '$t$' at the grid point $(x_l, t_j)$

$$U_{21} - U_{02} = V_{01} \tag{15.1}$$

$$V_{21} - V_{02} = G + HU_{01} + IV_{01} + JU_{11} + KV_{11}. \tag{15.2}$$

Using Taylor's series expansion, we first obtain

$$U_{l+1}^j - (1+\sigma_l)U_l^j + \sigma_l U_{l-1}^j = \frac{h_l^2}{12}[P_l(V_{l+1}^j + U_{t\,l+1}^j) + Q_l(V_l^j + U_{t\,l}^j) + R_l(V_{l-1}^j + U_{t\,l-1}^j)] + O(h_l^5), \tag{16.1}$$

$$V_{l+1}^j - (1+\sigma_l)V_l^j + \sigma_l V_{l-1}^j = \frac{h_l^2}{12}[P_l(F_{l+1}^j + V_{t\,l+1}^j) + Q_l(F_l^j + V_{t\,l}^j) + R_l(F_{l-1}^j + V_{t\,l-1}^j)] + O(h_l^5). \tag{16.2}$$

Simplifying the approximations (8.1)–(8.10), we obtain

$$\overline{U}_l^j = U_l^j + \theta k U_{01} + O(k^2), \tag{17.1}$$

$$\overline{U}_{l+1}^j = U_{l+1}^j + \theta[kU_{01} + k\sigma_l h_l U_{11}] + O(k^2 + kh_l^2), \tag{17.2}$$

$$\overline{U}_{l-1}^j = U_{l-1}^j + \theta[kU_{01} - kh_l U_{11}] + O(k^2 + kh_l^2), \tag{17.3}$$

$$\overline{V}_l^j = V_l^j + \theta k V_{01} + O(k^2), \tag{17.4}$$

$$\overline{V}_{l+1}^j = V_{l+1}^j + \theta[kV_{01} + k\sigma_l h_l V_{11}] + O(k^2 + kh_l^2), \tag{17.5}$$

$$\overline{V}_{l-1}^j = V_{l-1}^j + \theta[kV_{01} - kh_l V_{11}] + O(k^2 + kh_l^2), \tag{17.6}$$

$$\overline{U}_{t\,l}^j = U_{t\,l}^j + \frac{k}{2}U_{02} + O(k^2), \tag{17.7}$$

$$\overline{U}^j_{t\,l+1} = U^j_{t\,l+1} + \frac{k}{2}U_{02} + \frac{k\sigma_l h_l}{2}U_{12} + O(k^2 + kh_l^2), \tag{17.8}$$

$$\overline{U}^j_{t\,l-1} = U^j_{t\,l-1} + \frac{k}{2}U_{02} - \frac{kh_l}{2}U_{12} + O(k^2 + kh_l^2), \tag{17.9}$$

$$\overline{V}^j_{t\,l} = V^j_{t\,l} + \frac{k}{2}V_{02} + O(k^2), \tag{17.10}$$

$$\overline{V}^j_{t\,l+1} = V^j_{t\,l+1} + \frac{k}{2}V_{02} + \frac{k\sigma_l h_l}{2}V_{12} + O(k^2 + kh_l^2), \tag{17.11}$$

$$\overline{V}^j_{t\,l-1} = V^j_{t\,l-1} + \frac{k}{2}V_{02} - \frac{kh_l}{2}V_{12} + O(k^2 + kh_l^2), \tag{17.12}$$

$$\overline{U}^j_{xl} = U^j_{x\,l} + \frac{\sigma_l h_l^2}{6}U_{30} + \theta kU_{11} + O(k^2 + kh_l + h_l^3), \tag{17.13}$$

$$\overline{U}^j_{xl+1} = U^j_{x\,l+1} - \frac{\sigma_l(1+\sigma_l)h_l^2}{6}U_{30} + \theta kU_{11} + O(k^2 + kh_l + h_l^3), \tag{17.14}$$

$$\overline{U}^j_{xl-1} = U^j_{x\,l-1} - \frac{(1+\sigma_l)h_l^2}{6}U_{30} + \theta kU_{11} + O(k^2 + kh_l + h_l^3), \tag{17.15}$$

$$\overline{V}^j_{xl} = V^j_{x\,l} + \frac{\sigma_l h_l^2}{6}V_{30} + \theta kV_{11} + O(k^2 + kh_l + h_l^3), \tag{17.16}$$

$$\overline{V}^j_{xl+1} = V^j_{x\,l+1} - \frac{\sigma_l(1+\sigma_l)h_l^2}{6}V_{30} + \theta kV_{11} + O(k^2 + kh_l + h_l^3), \tag{17.17}$$

$$\overline{V}^j_{xl-1} = V^j_{x\,l-1} - \frac{(1+\sigma_l)h_l^2}{6}V_{30} + \theta kV_{11} + O(k^2 + kh_l + h_l^3). \tag{17.18}$$

By the help of approximations (17.1)–(17.18), from (9), we get

$$\overline{F}^j_{l+1} = F^j_{l+1} + \theta k[G + HU_{01} + IV_{01} + JU_{11} + KV_{11}] - \frac{\sigma_l(1+\sigma_l)h_l^2}{6}[JU_{30} + KV_{30}] + O(k^2 + kh_l + h_l^3), \tag{18.1}$$

$$\overline{F}^j_{l-1} = F^j_{l-1} + \theta k[G + HU_{01} + IV_{01} + JU_{11} + KV_{11}] - \frac{(1+\sigma_l)h_l^2}{6}[JU_{30} + KV_{30}] + O(k^2 + kh_l + h_l^3). \tag{18.2}$$

Using the approximations (17.1)–(17.18) in (10.1) and (10.2), we obtain

$$\overline{\overline{U}}^j_{xl} = U^j_{x\,l} + \theta kU_{11} + \frac{h_l^2}{6}[\sigma_l + 6a_l(1+\sigma_l)]U_{30} + O(k^2 + kh_l + h_l^3), \tag{19.1}$$

$$\overline{\overline{V}}^j_{xl} = V^j_{x\,l} + \theta kV_{11} + \frac{h_l^2}{6}[\sigma_l + 6b_l(1+\sigma_l)]V_{30} + O(k^2 + kh_l + h_l^3). \tag{19.2}$$

Finally, using approximations (7), (17.1), (17.4), (19.1), (19.2) in (11), we obtain

$$\begin{aligned}\overline{\overline{F}}^j_l &= F^j_l + \theta k[G + HU_{01} + IV_{01} + JU_{11} + KV_{11}] \\ &\quad + \frac{h_l^2}{6}[(\sigma_l + 6a_l(1+\sigma_l))JU_{30} + (\sigma_l + 6b_l(1+\sigma_l))KV_{30}] + O(k^2 + kh_l + h_l^3).\end{aligned} \tag{20}$$

Further, using the approximations (8.1), we may write

$$\overline{U}^j_{l+1} - (1+\sigma_l)\overline{U}^j_l + \sigma_l\overline{U}^j_{l-1} = [U^j_{l+1} - (1+\sigma_l)U^j_l + \sigma_l U^j_{l-1}] + \frac{\theta}{2}S_l kh_l^2 U_{21} + O(k^2h_l^2 + kh_l^3), \quad \sigma_l \neq 1. \tag{21}$$

Using the approximations (17.1)–(17.18) on the right hand side, (21) on the left hand side of (12.1) and by the help of relations (15.1) and (16.1), we obtain the local truncation error $\overline{T}_l^{j^{(1)}}$ associated with (12.1) as:

$$\overline{T}_l^{j^{(1)}} = kh_l^2\frac{S_l}{2}\left(\theta - \frac{1}{2}\right)U_{02} + O(k^2h_l^2 + kh_l^3 + h_l^5) \tag{22}$$

For the proposed difference method (12.1) to be of $O(k^2 + kh_l + h_l^3)$, the coefficient of $kh_l^2$ in (22) must be zero, thus we obtain $\theta = \frac{1}{2}$ and the local truncation error $\overline{T}_l^{j^{(1)}}$ reduces to $O(k^2h_l^2 + kh_l^3 + h_l^5)$. Also,

$$\overline{V}^j_{l+1} - (1+\sigma_l)\overline{V}^j_l + \sigma_l\overline{V}^j_{l-1} = [V^j_{l+1} - (1+\sigma_l)V^j_l + \sigma_l V^j_{l-1}] + \frac{\theta}{2}S_l kh_l^2 V_{21} + O(k^2h_l^2 + kh_l^3), \quad \sigma_l \neq 1. \tag{23}$$

Now using the approximations (17.10)–(17.12), (18.1)–(18.2), (20) on the right hand side, (23) on the left hand side of (12.2) and by the help of relations (15.2), (16.2), $\theta = \frac{1}{2}$, we obtain the local truncation error $\overline{T}_l^{j(2)}$ associated with (12.2) as:

$$\begin{aligned}\overline{T}_l^{j(2)} &= \frac{h_l^4}{72}[(-S_lP_l + (\sigma_l + 6a_l(1+\sigma_l))Q_l - (1+\sigma_l)R_l)JU_{30} + (-S_lP_l + (\sigma_l + 6b_l(1+\sigma_l))Q_l - (1+\sigma_l)R_l)KV_{30}] \\ &\quad + O(k^2h_l^2 + kh_l^3 + h_l^5), \quad \sigma_l \neq 1. \end{aligned} \tag{24}$$

The proposed difference method (12.2) to be of $O(k^2 + kh_l + h_l^3)$, the coefficient of $h_l^4$ in (24) must be zero, thus we obtain the values of parameters

$$a_l = b_l = \frac{-\sigma_l(1+\sigma_l+\sigma_l^2)}{6Q_l}$$

and the local truncation error associated with (12.2) reduces to $\overline{T}_l^{j(2)} = O(k^2h_l^2 + kh_l^3 + h_l^5)$, $\sigma_l \neq 1$ and thus the difference method (12.1)–(12.2) is of $O(k^2 + kh_l + h_l^3)$.

For $\sigma_l = 1$ (uniform mesh case), that is, for $h_{l+1} = h_l = h$, the method (12.1) and (12.2), for $\theta = \frac{1}{2}$ reduces to the following two-level implicit difference method of $O(k^2 + kh^2 + h^4)$ for the solution of differential Eqs. (4.1) and (4.2):

$$\delta_x^2\overline{U}_l^j = \frac{h^2}{12}[(\overline{V}_{l+1}^j + \overline{U}_{t\,l+1}^j) + 10(\overline{V}_l^j + \overline{U}_{t\,l}^j) + (\overline{V}_{l-1}^j + \overline{U}_{t\,l-1}^j)] + O(k^2h^2 + kh^4 + h^6), \tag{25.1}$$

$$\delta_x^2\overline{V}_l^j = \frac{h^2}{12}[(\overline{F}_{l+1}^j + \overline{V}_{t\,l+1}^j) + 10(\overline{\overline{F}}_l^j + \overline{V}_{t\,l}^j) + (\overline{F}_{l-1}^j + \overline{V}_{t\,l-1}^j)] + O(k^2h^2 + kh^4 + h^6), \tag{25.2}$$

where $\delta_x$ is the central difference operator with respect to $x$-direction.

## 3. Three level implicit method

Consider another type of fourth-order quasi-linear parabolic PDE in one-space dimension having the form

$$A(x,t,u,u_{xx})\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = g(x,t,u,u_t,u_x,u_{xx},u_{xxx}), \quad a < x < b, \quad t > 0 \tag{26}$$

or equivalently,

$$\frac{\partial^2 u}{\partial x^2} = v, \quad (x,t) \in \Omega \tag{27.1}$$

$$A(x,t,u,v)\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = g(x,t,u,v,u_t,u_x,v_x), \quad (x,t) \in \Omega \tag{27.2}$$

subject to the initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad a \le x \le b \tag{28}$$

where $u_0$ and $u_1$ are sufficiently differentiable functions and the boundary conditions are given by

$$\begin{aligned} &u(a,t) = g_0(t), \quad u(b,t) = g_1(t), \quad t > 0 \\ &u_{xx}(a,t) = h_0(t), \quad u_{xx}(b,t) = h_1(t), \quad t > 0 \end{aligned} \tag{29}$$

or equivalently,

$$v(a,t) = h_0(t), \; v(b,t) = h_1(t), \quad t > 0 \tag{30}$$

Note that the boundary values of $u$ and $v$ are prescribed at $x = a$ and $x = b$ and the initial values of $u$ and $u_t$ are provided at $t = 0$. Since the value of $u$ is known explicitly at $t = 0$, this implies all the successive tangential partial derivatives of $u$ are known at $t = 0$, i.e. the values of $u_x$, $u_{xx}$, ..., etc. are known at $t = 0$. As $v(x,0) = u_{xx}(x,0)$, so the value of $v$ is also known at $t = 0$.

For simplicity, we first consider the fourth-order non-linear parabolic PDE of the form:

$$A(x,t)\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = g(x,t,u,u_t,u_x,u_{xx},u_{xxx}), \quad a < x < b, \quad t > 0 \tag{31}$$

or equivalently,

$$\frac{\partial^2 u}{\partial x^2} = v, \quad (x,t) \in \Omega \tag{32.1}$$

$$A(x,t)\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = g(x,t,u,v,u_t,u_x,v_x), \quad (x,t) \in \Omega \tag{32.2}$$

subject to the initial and boundary conditions given by (28)–(30).

To determine the numerical solution of the proposed initial boundary value problem (32.1) and (32.2) together with (28)–(30), we discretize the solution domain $\Omega$ using the variable mesh defined in Section 2 and at the grid point $(x_l, t_j)$, we denote

$$P_{1_l} = \frac{(\sigma_l - 1)}{3} - \frac{h_l}{18}(1+\sigma_l+\sigma_l^2)\frac{A^j_{x_l}}{A^j_l} \tag{33.1}$$

$$P_{2_l} = \frac{1-\sigma_l+\sigma_l^2}{12},$$

$$P^*_l = \sigma_l^2+\sigma_l-1-\frac{h_l}{3}(1+\sigma_l+\sigma_l^2)\frac{A^j_{x_l}}{A^j_l}, \tag{33.2}$$

$$Q^*_l = (1+\sigma_l)(1+3\sigma_l+\sigma_l^2)+\frac{h_l}{3}(1-\sigma_l^2)(1+\sigma_l+\sigma_l^2)\frac{A^j_{x_l}}{A^j_l}, \tag{33.3}$$

$$R^*_l = \sigma_l(1+\sigma_l-\sigma_l^2)+\frac{h_l}{3}\sigma_l^2(1+\sigma_l+\sigma_l^2)\frac{A^j_{x_l}}{A^j_l} \tag{33.4}$$

and $P_l, Q_l, R_l, S_l$ are as defined in (6).

Our variable mesh numerical methods are described as follows. For $p = 0, \pm 1$; let

$$\widehat{U}^j_{l+p} = \tau U^{j+1}_{l+p} + (1-2\tau)U^j_{l+p} + \tau U^{j-1}_{l+p}, \quad 0<\tau<1 \tag{34.1}$$

$$\widehat{V}^j_{l+p} = \tau V^{j+1}_{l+p} + (1-2\tau)V^j_{l+p} + \tau V^{j-1}_{l+p}, \quad 0<\tau<1 \tag{34.2}$$

$$\widehat{U}^j_{t_{l+p}} = (U^{j+1}_{l+p} - U^{j-1}_{l+p})/(2k), \tag{34.3}$$

$$\widehat{U}^j_{tt_{l+p}} = (U^{j+1}_{l+p} - 2U^j_{l+p} + U^{j-1}_{l+p})/k^2, \tag{34.4}$$

$$\widehat{U}^j_{x_l} = (\widehat{U}^j_{l+1} - (1-\sigma_l^2)\widehat{U}^j_l - \sigma_l^2\widehat{U}^j_{l-1})/(h_lS_l), \tag{34.5}$$

$$\widehat{U}^j_{x_{l+1}} = ((1+2\sigma_l)\widehat{U}^j_{l+1} - (1+\sigma_l)^2\widehat{U}^j_l + \sigma_l^2\widehat{U}^j_{l-1})/(h_lS_l), \tag{34.6}$$

$$\widehat{U}^j_{x_{l-1}} = (-\widehat{U}^j_{l+1} + (1+\sigma_l)^2\widehat{U}^j_l - \sigma_l(2+\sigma_l)\widehat{U}^j_{l-1})/(h_lS_l), \tag{34.7}$$

$$\widehat{V}^j_{x_l} = (\widehat{V}^j_{l+1} - (1-\sigma_l^2)\widehat{V}^j_l - \sigma_l^2\widehat{V}^j_{l-1})/(h_lS_l), \tag{34.8}$$

$$\widehat{V}^j_{x_{l+1}} = ((1+2\sigma_l)\widehat{V}^j_{l+1} - (1+\sigma_l)^2\widehat{V}^j_l + \sigma_l^2\widehat{V}^j_{l-1})/(h_lS_l), \tag{34.9}$$

$$\widehat{V}^j_{x_{l-1}} = (-\widehat{V}^j_{l+1} + (1+\sigma_l)^2\widehat{V}^j_l - \sigma_l(2+\sigma_l)\widehat{V}^j_{l-1})/(h_lS_l), \tag{34.10}$$

$$\widehat{V}^j_{xx_l} = 2(\widehat{V}^j_{l+1} - (1+\sigma_l)\widehat{V}^j_l + \sigma_l\widehat{V}^j_{l-1})/(h_l^2S_l), \tag{34.11}$$

$$\widehat{G}^j_{l+p} = g(x_{l+p}, t_j, \widehat{U}^j_{l+p}, \widehat{V}^j_{l+p}, \widehat{U}^j_{t_{l+p}}, \widehat{U}^j_{x_{l+p}}, \widehat{V}^j_{x_{l+p}}), \tag{34.12}$$

$$\widehat{\widehat{U}}^j_{x_l} = \widehat{U}^j_{x_l} + p_1h_l[\widehat{V}^j_{l+1} - \widehat{V}^j_{l-1}], \tag{34.13}$$

$$\widehat{\widehat{V}}^j_{x_l} = \widehat{V}^j_{x_l} + q_1h_l[(\widehat{G}^j_{l+1} - \widehat{G}^j_{l-1}) - (\widehat{U}^j_{tt_{l+1}} - \widehat{U}^j_{tt_{l-1}})] + q_2h_l^2\widehat{V}^j_{xx_l}, \tag{34.14}$$

$$\widehat{\widehat{G}}^j_l = g(x_l, t_j, \widehat{U}^j_l, \widehat{V}^j_l, \widehat{U}^j_{t_l}, \widehat{\widehat{U}}^j_{x_l}, \widehat{\widehat{V}}^j_{x_l}), \tag{34.15}$$

where $p_1$, $q_1$ and $q_2$ are the parameters to be suitably determined. Then the difference methods of $O(k^2+h_l^3)$ for the differential Eqs. (32.1) and (32.2) are given by

$$\widehat{U}^j_{l+1} - (1+\sigma_l)\widehat{U}^j_l + \sigma_l\widehat{U}^j_{l-1} = \frac{h_l^2}{12}[P_l\widehat{V}^j_{l+1} + Q_l\widehat{V}^j_l + R_l\widehat{V}^j_{l-1}] + \widehat{T}^{j^{(1)}}_l, \tag{35.1}$$

$$\begin{aligned}&[A^j_l + h_lP_{1_l}A^j_{x_l} + h_l^2P_{2_l}A^j_{xx_l}][\widehat{V}^j_{l+1} - (1+\sigma_l)\widehat{V}^j_l + \sigma_l\widehat{V}^j_{l-1}]\\ &\quad = \frac{h_l^2}{12}[P^*_l(\widehat{G}^j_{l+1} - \widehat{U}^j_{tt_{l+1}}) + Q^*_l(\widehat{\widehat{G}}^j_l - \widehat{U}^j_{tt_l}) + R^*_l(\widehat{G}^j_{l-1} - \widehat{U}^j_{tt_{l-1}})] + \widehat{T}^{j^{(2)}}_l,\end{aligned} \tag{35.2}$$

where $\widehat{T}_l^{j(1)} = O(k^2h_l^3 + h_l^5)$, $\widehat{T}_l^{j(2)} = O(k^2h_l^2 + k^2h_l^3 + h_l^5)$, provided $\sigma_l \neq 1$.

For the derivation of the numerical method (35.1) and (35.2), we follow the techniques given by Mohanty and Evans [16,23]. At the grid point $(x_l, t_j)$, we write

$$G_l^j = g(x_l, t_j, U_l^j, V_l^j, U_{t_l}^j, U_{x_l}^j, V_{x_l}^j),$$
$$G_{l\pm1}^j = g(x_{l\pm1}, t_j, U_{l\pm1}^j, V_{l\pm1}^j, U_{t_{l\pm1}}^j, U_{x_{l\pm1}}^j, V_{x_{l\pm1}}^j).$$

In the following in this section, at the grid point $(x_l, t_j)$, we let

$$\alpha = \frac{\partial g}{\partial u}, \quad \beta = \frac{\partial g}{\partial v}, \quad \gamma = \frac{\partial g}{\partial u_x}, \quad \delta = \frac{\partial g}{\partial v_x}, \quad \zeta = \frac{\partial g}{\partial u_t}.$$

Simplifying the approximations (34.1)–(34.11), we obtain

$$\widehat{U}_l^j = U_l^j + \tau k^2 U_{02} + O(k^4), \tag{36.1}$$

$$\widehat{U}_{l\pm1}^j = U_{l\pm1}^j + \tau k^2 U_{02} + O(\pm h_l k^2 + h_l^2 k^2), \tag{36.2}$$

$$\widehat{V}_l^j = V_l^j + \tau k^2 V_{02} + O(k^4), \tag{36.3}$$

$$\widehat{V}_{l\pm1}^j = V_{l\pm1}^j + \tau k^2 V_{02} + O(\pm h_l k^2 + h_l^2 k^2), \tag{36.4}$$

$$\widehat{U}_{t_l}^j = U_{t_l}^j + \frac{k^2}{6} U_{03} + O(k^4), \tag{36.5}$$

$$\widehat{U}_{t_{l\pm1}}^j = U_{t_{l\pm1}}^j + \frac{k^2}{6} U_{03} + O(\pm h_l k^2 + h_l^2 k^2), \tag{36.6}$$

$$\widehat{U}_{tt_l}^j = U_{tt_l}^j + \frac{k^2}{12} U_{04} + O(k^4), \tag{36.7}$$

$$\widehat{U}_{tt_{l\pm1}}^j = U_{tt_{l\pm1}}^j + \frac{k^2}{12} U_{04} + O(\pm h_l k^2 + h_l^2 k^2), \tag{36.8}$$

$$\widehat{U}_{x_l}^j = U_{x_l}^j + \frac{\sigma_l h_l^2}{6} U_{30} + \tau k^2 U_{12} + O(h_l^3 + h_l^2 k^2), \tag{36.9}$$

$$\widehat{U}_{x_{l+1}}^j = U_{x_{l+1}}^j - \frac{\sigma_l(1+\sigma_l)h_l^2}{6} U_{30} + \tau k^2 U_{12} + O(h_l^3 + h_l k^2), \tag{36.10}$$

$$\widehat{U}_{x_{l-1}}^j = U_{x_{l-1}}^j - \frac{(1+\sigma_l)h_l^2}{6} U_{30} + \tau k^2 U_{12} + O(h_l^3 - h_l k^2), \tag{36.11}$$

$$\widehat{V}_{x_l}^j = V_{x_l}^j + \frac{\sigma_l h_l^2}{6} V_{30} + \tau k^2 V_{12} + O(h_l^3 + h_l^2 k^2), \tag{36.12}$$

$$\widehat{V}_{x_{l+1}}^j = V_{x_{l+1}}^j - \frac{\sigma_l(1+\sigma_l)h_l^2}{6} V_{30} + \tau k^2 V_{12} + O(h_l^3 + h_l k^2), \tag{36.13}$$

$$\widehat{V}_{x_{l-1}}^j = V_{x_{l-1}}^j - \frac{(1+\sigma_l)h_l^2}{6} V_{30} + \tau k^2 V_{12} + O(h_l^3 - h_l k^2), \tag{36.14}$$

$$\widehat{V}_{xx_l}^j = V_{xx_l}^j + O(k^2 + h_l). \tag{36.15}$$

Using (36.1)–(36.14) in (34.12), we get

$$\begin{aligned}\widehat{G}_{l+1}^j = G_{l+1}^j &+ \frac{k^2}{6}[6\tau(\alpha U_{02} + \beta V_{02} + \gamma U_{12} + \delta V_{12}) + \zeta U_{03}] \\ &- \frac{\sigma_l(1+\sigma_l)h_l^2}{6}[\gamma U_{30} + \delta V_{30}] + O(h_l^3 + h_l k^2 + h_l^2 k^2),\end{aligned} \tag{37.1}$$

$$\begin{aligned}\widehat{G}_{l-1}^j = G_{l-1}^j &+ \frac{k^2}{6}[6\tau(\alpha U_{02} + \beta V_{02} + \gamma U_{12} + \delta V_{12}) + \zeta U_{03}] \\ &- \frac{(1+\sigma_l)h_l^2}{6}[\gamma U_{30} + \delta V_{30}] + O(h_l^3 - h_l k^2 + h_l^2 k^2).\end{aligned} \tag{37.2}$$

Next, by the help of approximations (36.1)–(36.15), (37.1) and (37.2), from (34.13) and (34.14), we obtain

$$\widehat{\widehat{U}}_{x_l}^j = U_{x_l}^j + \tau k^2 U_{12} + \frac{h_l^2}{6}[\sigma_l + 6p_1(1+\sigma_l)]U_{30} + O(h_l^3 + h_l^2 k^2), \tag{38.1}$$

$$\widehat{\widehat{V}}_{x_l}^j = V_{x_l}^j + \tau k^2 V_{12} + \frac{h_l^2}{6}[(\sigma_l + 6q_1(1+\sigma_l)A_l^j)V_{30} + 6(q_1(1+\sigma_l)A_{x_l}^j + q_2)V_{20}] + O(h_l^3 + h_l k^2 + h_l^2 k^2). \tag{38.2}$$

Finally, by using (36.1)–(36.5), (38.1) and (38.2), from (34.15) we get

$$\begin{aligned}\widehat{\widehat{G}}_l^j &= G_l^j + \frac{k^2}{6}[6\tau(\alpha U_{02} + \beta V_{02} + \gamma U_{12} + \delta V_{12}) + \zeta U_{03}] \\ &\quad + \frac{h_l^2}{6}[(\sigma_l + 6p_1(1+\sigma_l))\gamma U_{30} + (\sigma_l + 6q_1(1+\sigma_l)A_l^j)\delta V_{30} + 6(q_1(1+\sigma_l)A_{x_l}^j + q_2)\delta V_{20}] + O(h_l^3 + h_l k^2 + h_l^2 k^2)\end{aligned} \tag{39}$$

By the help of Taylor's series expansion we obtain

$$\widehat{U}_{l+1}^j - (1+\sigma_l)\widehat{U}_l^j + \sigma_l \widehat{U}_{l-1}^j = [U_{l+1}^j - (1+\sigma_l)U_l^j + \sigma_l U_{l-1}^j] + \tau S_l \frac{h_l^2 k^2}{2} U_{22} + O(k^2 h_l^3 + k^4), \tag{40.1}$$

$$\begin{aligned}&[A_l^j + h_l P_{1_l} A_{x_l}^j + h_l^2 P_{2_l} A_{xx_l}^j][\widehat{V}_{l+1}^j - (1+\sigma_l)\widehat{V}_l^j + \sigma_l \widehat{V}_{l-1}^j] \\ &= [A_l^j + h_l P_{1_l} A_{x_l}^j + h_l^2 P_{2_l} A_{xx_l}^j][V_{l+1}^j - (1+\sigma_l)V_l^j + \sigma_l V_{l-1}^j] + \tau S_l \frac{h_l^2 k^2}{2} A_{00} V_{22} + O(k^2 h_l^3 + k^4), \quad \sigma_l \neq 1.\end{aligned} \tag{40.2}$$

Also, we have

$$U_{l+1}^j - (1+\sigma_l)U_l^j + \sigma_l U_{l-1}^j = \frac{h_l^2}{12}[P_l V_{l+1}^j + Q_l V_l^j + R_l V_{l-1}^j] + O(h_l^5), \tag{41.1}$$

$$\begin{aligned}&[A_l^j + h_l P_{1_l} A_{x_l}^j + h_l^2 P_{2_l} A_{xx_l}^j][V_{l+1}^j - (1+\sigma_l)V_l^j + \sigma_l V_{l-1}^j] \\ &= \frac{h_l^2}{12}[P_l^*(G_{l+1}^j - U_{tt_{l+1}}^j) + Q_l^*(G_l^j - U_{tt_l}^j) + R_l^*(G_{l-1}^j - U_{tt_{l-1}}^j)] + O(h_l^5).\end{aligned} \tag{41.2}$$

Since $U_{22} = V_{02}$, using (36.3), (36.4) and (40.1) in (35.1) and by (41.1), we conclude that the local truncation error $\widehat{T}_l^{j(1)}$ associated with (35.1) may be obtained as $\widehat{T}_l^{j(1)} = O(k^2 h_l^3 + h_l^5)$ for arbitrary $\tau$. Similarly, using the approximations (36.1)–(36.15), (37.1), (37.2) and (39) in (35.2) and by the help of relations (40.2), (41.2), we obtain the local truncation error $\widehat{T}_l^{j(2)}$ associated with (35.2) as:

$$\begin{aligned}\widehat{T}_l^{j(2)} &= \frac{h_l^4}{72}\Big[(-S_l P_l^* + (\sigma_l + 6p_1(1+\sigma_l))Q_l^* - (1+\sigma_l)R_l^*)\gamma U_{30} + (-S_l P_l^* + (\sigma_l + 6q_1(1+\sigma_l)A_l^j)Q_l^* - (1+\sigma_l)R_l^*)\delta V_{30} \\ &\quad + 6Q_l^*(q_1(1+\sigma_l)A_{x_l}^j + q_2)\delta V_{20}\Big] + \frac{h_l^2 k^2}{2} S_l \left[\frac{1}{6}(6\tau(\alpha U_{02} + \beta V_{02} + \gamma U_{12} + \delta V_{12} - A_{00}V_{22}) + \zeta U_{03}) - \frac{1}{12}U_{04}\right] \\ &\quad + O(k^2 h_l^3 + h_l^5),\end{aligned} \tag{42}$$

provided $\sigma_l \neq 1$. The proposed difference method (35.2) to be of $O(k^2 + h_l^3)$, the coefficient of $h_l^4$ in (42) must be zero, hence we obtain the values of the parameters

$$p_1 = \frac{-\sigma_l(1+\sigma_l+\sigma_l^2)}{6Q_l^*}\left(1 + \frac{2(1-\sigma_l)h_l A_{x_l}^j}{3A_l^j}\right),$$

$$q_1 = \frac{-\sigma_l(1+\sigma_l+\sigma_l^2)}{6A_l^j Q_l^*}\left(1 + \frac{2(1-\sigma_l)h_l A_{x_l}^j}{3A_l^j}\right),$$

$$q_2 = \frac{\sigma_l(1+\sigma_l)(1+\sigma_l+\sigma_l^2)A_{x_l}^j}{6A_l^j Q_l^*}\left(1 + \frac{2(1-\sigma_l)h_l A_{x_l}^j}{3A_l^j}\right)$$

and thus the local truncation error (42) reduces to $\widehat{T}_l^{j(2)} = O(k^2 h_l^2 + k^2 h_l^3 + h_l^5)$, provided $\sigma_l \neq 1$, for arbitrary $\tau$.

For the numerical solution of quasi-linear problem in the coupled form given by Eqs. (27.1) and (27.2) subject to initial and boundary conditions (28)–(30), i.e. when the coefficient $A = A(x, t, u, v)$, then we need to modify our proposed difference method (35.1) and (35.2). In this case, we make use of the following approximations in (35.1) and (35.2):

$$A_{x_l}^j = \frac{\overline{A}_{l+1}^j - (1-\sigma_l^2)\overline{A}_l^j - \sigma_l^2 \overline{A}_{l-1}^j}{h_l S_l}, \tag{43.1}$$

$$A^j_{xx_l} = \frac{2(\overline{A}^j_{l+1} - (1+\sigma_l)\overline{A}^j_l + \sigma_l \overline{A}^j_{l-1})}{h_l^2 S_l}, \tag{43.2}$$

where

$$\overline{A}^j_l = A(x_l, t_j, \overline{U}^j_l, \overline{V}^j_l), \tag{43.3}$$

$$\overline{A}^j_{l\pm1} = A(x_{l\pm1}, t_j, \overline{U}^j_{l\pm1}, \overline{V}^j_{l\pm1}). \tag{43.4}$$

Using the approximations (43.1)-(43.4) and proceeding in the same way as above, we obtain difference method of $O(k^2 + h_l^3)$ for the numerical solution of quasi-linear Eqs. (27.1)–(27.2).

For $\sigma_l = 1$ (uniform mesh case), that is, for $h_{l+1} = h_l = h$, the method (35.1) and (35.2), for arbitrary $\tau$ reduces to the following three-level implicit difference method of $O(k^2 + h^4)$ for the solution of differential equations (32.1) and (32.2) (see Mohanty and Evans [16]):

$$\delta_x^2 \overline{U}^j_l = \frac{h^2}{12}[\overline{V}^j_{l+1} + 10\overline{V}^j_l + \overline{V}^j_{l-1}] + O(k^2h^4 + h^6), \tag{44.1}$$

$$\left[A_{00} + \frac{h^2}{12}\left(A_{20} - \frac{2A_{10}^2}{A_{00}}\right)\right]\delta_x^2 \overline{V}^j_l = \frac{h^2}{12}\left[\left(1 - \frac{hA_{10}}{A_{00}}\right)(\overline{G}^j_{l+1} - \overline{U}^j_{tt_{l+1}}) + 10(\overline{\overline{G}}^j_l - \overline{U}^j_{tt_l}) + \left(1 + \frac{hA_{10}}{A_{00}}\right)(\overline{G}^j_{l-1} - \overline{U}^j_{tt_{l-1}})\right]$$
$$+ O(k^2h^2 + k^2h^4 + h^6). \tag{44.2}$$

## 4. Stability consideration and application to linear singular problem

We shall discuss the stability for the uniform mesh case. For simplicity, we consider the fourth-order parabolic PDE of the form

$$\frac{\partial^4 u}{\partial x^4} - 2\frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial^2 u}{\partial t^2} = f(x,t), \quad a < x < b, \ \ t > 0 \tag{45}$$

The finite difference method (25.1) and (25.2) applied to model equation (45) can be written in the matrix form as:

$$AY^{j+1} = BY^j + l \tag{46}$$

where

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_1 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_2 \\ 0 & B_1 \end{bmatrix}, \quad Y = \begin{bmatrix} u \\ v \end{bmatrix}, \quad l = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix},$$

$$A_1 = [1, 10, 1] - 6\lambda[1, -2, 1], \quad A_2 = \frac{k}{2}[1, 10, 1],$$

$$B_1 = [1, 10, 1] + 6\lambda[1, -2, 1], \quad B_2 = \frac{-k}{2}[1, 10, 1],$$

where $\lambda = k/h^2$ is the mesh ratio parameter for the uniform mesh (for $\sigma_l = 1$, that is, for $h_{l+1} = h_l = h$) and we denote

$$[c, d, e] = \begin{bmatrix} d & e & 0 & \cdots & 0 \\ c & d & e & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & c & d & e \\ 0 & \cdots & 0 & c & d \end{bmatrix}$$

as $N$th-order tridiagonal matrix whose eigenvalues are given by $d + 2\sqrt{ce}\cos(2\phi)$, $\phi = (s\pi)/(2(N+1))$, $s = 1(1)N$. $u$, $v$ are solution vectors and vectors $l_1, l_2$ consist of homogenous functions, initial and boundary values of the block system (46).

For stability, we consider the homogenous part of (46), which may be written as

$$Y^{j+1} = (A^{-1}B)Y^j \tag{47}$$

We denote $\varepsilon^j = y^j - Y^j$ as the error vector at the $j$th iterate (in the absence of round-off errors), where

$$Y^j = \begin{bmatrix} U \\ V \end{bmatrix}^j, \quad Y^{j+1} = \begin{bmatrix} U \\ V \end{bmatrix}^{j+1}$$

The error equation can be written as

$$\varepsilon^{j+1} = H\varepsilon^j$$

where $H = A^{-1}B$ is the amplification matrix. The eigenvalues of $A_1$ and $A_2$ are given by $12 - 4\sin^2\phi + 24\lambda\sin^2\phi$ and $\frac{k}{2}(12 - 4\sin^2\phi)$, respectively while the eigenvalues of $B_1$ and $B_2$ are given by $12 - 4\sin^2\phi - 24\lambda\sin^2\phi$ and $\frac{-k}{2}(12 - 4\sin^2\phi)$, respectively. Hence, the eigenvalues of matrices $A$ and $B$ for the difference method (25.1) and (25.2) are given by $12 - 4\sin^2\phi + 24\lambda\sin^2\phi$ and $12 - 4\sin^2\phi - 24\lambda\sin^2\phi$, respectively.

Let $\nu$ be the eigenvalue of $A^{-1}B$. Since the matrices $A^{-1}$ and $B$ commute each other, hence the eigenvalues $\nu$ are given by

$$\nu = \frac{12 - 4\sin^2\phi - 24\lambda\sin^2\phi}{12 - 4\sin^2\phi + 24\lambda\sin^2\phi} \tag{48}$$

Since $0 \le \sin^2\phi \le 1$, from (48), it is easy to verify that $|\nu| \le 1$ for all variable angle $\phi$ and $\lambda > 0$. Hence the method (25.1) and (25.2) is unconditionally stable for the differential Eq. (45).

For the stability of difference method (44.1) and (44.2) refer [16].

We consider the proposed difference method (35.1) and (35.2), on a variable mesh applied to a class of linear singular fourth-order parabolic equation in one-space dimension of the form:

$$\nabla^4 u + \frac{\partial^2 u}{\partial t^2} \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{\alpha}{r}\frac{\partial}{\partial r}\right)^2 u + \frac{\partial^2 u}{\partial t^2} = g(r,t), \quad 0 < r < 1, \quad t > 0$$

or equivalently,

$$\frac{\partial^4 u}{\partial r^4} + \frac{\partial^2 u}{\partial t^2} = B(r)\frac{\partial^3 u}{\partial r^3} + C(r)\frac{\partial^2 u}{\partial r^2} + D(r)\frac{\partial u}{\partial r} + g(r,t), \quad 0 < r < 1, \quad t > 0 \tag{49}$$

where

$$B(r) = \frac{-2\alpha}{r}, \quad C(r) = \frac{\alpha(2-\alpha)}{r^2}, \quad D(r) = \frac{\alpha(\alpha-2)}{r^3}.$$

For $\alpha = 1$ and 2,

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{\alpha}{r}\frac{\partial}{\partial r}$$

represents one-space dimensional Laplacian operator in cylindrical and spherical coordinates, respectively. Using the second order central differences, the numerical solution of (49) can be obtained making use of five grid points in $r$-direction and three grid points in $t$-direction. But so far, no difference method on a variable mesh of $O(k^2 + h_l^3)$ using only three grid points in both $r$- and $t$-directions is known for singular Eq. (49). The finite difference method discussed here is three-level in time based only on three spatial grid points and no fictitious points are taken for incorporating the boundary conditions. It is difficult to determine the numerical solution of (49) in polar coordinates as the solution deteriorates in the vicinity of the singularity $r = 0$. We overcome this difficulty by modifying our method in such a manner that the order and accuracy of the solution is retained everywhere including the region in the vicinity of the singularity $r = 0$.

Replacing the variable $x$ by $r$ and applying the difference method (35.1) and (35.2) to the singular Eq. (49), we obtain the following difference method upon neglecting the error terms

$$\widehat{u}_{l+1}^j - (1+\sigma_l)\widehat{u}_l^j + \sigma_l\widehat{u}_{l-1}^j = \frac{h_l^2}{12}[P_l\widehat{v}_{l+1}^j + Q_l\widehat{v}_l^j + R_l\widehat{v}_{l-1}^j], \tag{50.1}$$

$$\begin{aligned}
&[\widehat{v}_{l+1}^j - (1+\sigma_l)\widehat{v}_l^j + \sigma_l\widehat{v}_{l-1}^j] + \frac{h_l^2}{12k^2}\delta_t^2[(P_l - p_0)u_{l+1}^j + (R_l + p_0)u_{l-1}^j + Q_l u_l^j] \\
&= \frac{h_l^2}{12}\Big[Q_l B_l\widehat{v}_{r_l}^j + (P_l - p_0)B_{l+1}\widehat{v}_{r_{l+1}}^j + (R_l + p_0)B_{l-1}\widehat{v}_{r_{l-1}}^j + Q_l D_l\widehat{u}_{r_l}^j + (P_l - p_0)D_{l+1}\widehat{u}_{r_{l+1}}^j + (R_l + p_0)D_{l-1}\widehat{u}_{r_{l-1}}^j \\
&\quad + Q_l C_l\widehat{v}_l^j + (P_l - p_0)C_{l+1}\widehat{v}_{l+1}^j + (R_l + p_0)C_{l-1}\widehat{v}_{l-1}^j - q_0(\widehat{v}_{l+1}^j - \widehat{v}_{l-1}^j)\Big] \\
&\quad + \frac{h_l^2}{12}[Q_l g_l^j + (P_l - p_0)g_{l+1}^j + (R_l + p_0)g_{l-1}^j], \quad l = 1(1)N, \quad j = 1, 2, \ldots
\end{aligned} \tag{50.2}$$

where $\delta_t$ is the central difference operator with respect to $t$-direction, $p_0 = (\sigma_l(1+\sigma_l+\sigma_l^2)h_l B_l)/6$, $q_0 = (\sigma_l(1+\sigma_l+\sigma_l^2)h_l D_l)/6$ and for $p = 0, \pm 1$: $B_{l+p} = B(r_{l+p})$, $C_{l+p} = C(r_{l+p})$, $D_{l+p} = D(r_{l+p})$, $g_{l+p}^j = g(r_{l+p}, t_j)$.

Although the difference method (50.1) and (50.2) is of $O(k^2 + h_l^3)$ but it contain the terms $B_{l-1}$, $C_{l-1}$, $D_{l-1}$, $g_{l-1}^j$, all of which involves the term $1/r_{l-1}$ giving rise to singularity at $l = 1$ since $r_0 = 0$. It is for this reason that the proposed method is not directly applicable to singular equations in the region $0 < r < 1$, i.e., in the vicinity of singularity. We overcome this difficulty by modifying the method (50.2) in such a manner that the order and accuracy of the solution is retained everywhere in the region $[0 < r < 1] \times [t > 0]$, even in the vicinity of the singularity $r = 0$.

In order to get a valid difference method of $O(k^2 + h_l^3)$ on a variable mesh, we require the following approximations:

$$B_{l+1} = B_{00} + \sigma_l h_l B_{10} + \frac{\sigma_l^2 h_l^2}{2}B_{20} + O(h_l^3) \equiv B_1 + O(h_l^3), \tag{51.1}$$

$$B_{l-1} = B_{00} - h_l B_{10} + \frac{h_l^2}{2} B_{20} - O(h_l^3) \equiv B_2 - O(h_l^3), \tag{51.2}$$

$$C_{l+1} = C_{00} + \sigma_l h_l C_{10} + \frac{\sigma_l^2 h_l^2}{2} C_{20} + O(h_l^3) \equiv C_1 + O(h_l^3), \tag{51.3}$$

$$C_{l-1} = C_{00} - h_l C_{10} + \frac{h_l^2}{2} C_{20} - O(h_l^3) \equiv C_2 - O(h_l^3), \tag{51.4}$$

$$D_{l+1} = D_{00} + \sigma_l h_l D_{10} + \frac{\sigma_l^2 h_l^2}{2} D_{20} + O(h_l^3) \equiv D_1 + O(h_l^3), \tag{51.5}$$

$$D_{l-1} = D_{00} - h_l D_{10} + \frac{h_l^2}{2} D_{20} - O(h_l^3) \equiv D_2 - O(h_l^3), \tag{51.6}$$

$$g_{l+1}^j = g_{00} + \sigma_l h_l g_{10} + \frac{\sigma_l^2 h_l^2}{2} g_{20} + O(h_l^3) \equiv G_1 + O(h_l^3), \tag{51.7}$$

$$g_{l-1}^j = g_{00} - h_l g_{10} + \frac{h_l^2}{2} g_{20} - O(h_l^3) \equiv G_2 - O(h_l^3). \tag{51.8}$$

where

$$B_1 = B_{00} + \sigma_l h_l B_{10} + \frac{\sigma_l^2 h_l^2}{2} B_{20}, \quad B_2 = B_{00} - h_l B_{10} + \frac{h_l^2}{2} B_{20}, \text{ etc.}$$

Using the approximations (51.1)–(51.8) in the difference method (50.1) and (50.2) and neglecting the higher-order terms, we obtain

$$\widehat{u}_{l+1}^j - (1+\sigma_l)\widehat{u}_l^j + \sigma_l \widehat{u}_{l-1}^j = \frac{h_l^2}{12}[P_l \widehat{v}_{l+1}^j + Q_l \widehat{v}_l^j + R_l \widehat{v}_{l-1}^j], \tag{52.1}$$

$$\begin{aligned}
&[\widehat{v}_{l+1}^j - (1+\sigma_l)\widehat{v}_l^j + \sigma_l \widehat{v}_{l-1}^j] + \frac{h_l^2}{12k^2}\delta_t^2[(P_l - p_0)u_{l+1}^j + (R_l + p_0)u_{l-1}^j + Q_l u_l^j] \\
&= \frac{h_l^2}{12}\Big[Q_l B_{00}\widehat{v}_{r_l}^j + (P_l - p_0)B_1 \widehat{v}_{r_{l+1}}^j + (R_l + p_0)B_2 \widehat{v}_{r_{l-1}}^j + Q_l D_{00}\widehat{u}_{r_l}^j + (P_l - p_0)D_1 \widehat{u}_{r_{l+1}}^j + (R_l + p_0)D_2 \widehat{u}_{r_{l-1}}^j \\
&\quad + Q_l C_{00}\widehat{v}_l^j + (P_l - p_0)C_1 \widehat{v}_{l+1}^j + (R_l + p_0)C_2 \widehat{v}_{l-1}^j - q_0(\widehat{v}_{l+1}^j - \widehat{v}_{l-1}^j)\Big] \\
&\quad + \frac{h_l^2}{12}[Q_l g_{00} + (P_l - p_0)G_1 + (R_l + p_0)G_2], \quad l = 1(1)N, \quad j = 1, 2, \ldots
\end{aligned} \tag{52.2}$$

Note that the difference method (52.1) and (52.2) is a three-level implicit method of $O(k^2 + h_l^3)$ which does not involve the terms $1/(l \pm 1)$, hence can be very easily solved for $l = 1(1)N$ in the region $[0 < r < 1] \times [t > 0]$. The resulting system of tridiagonal matrices can be solved using block iterative methods (see Kelly [24]).

## 5. Numerical illustrations

For

$$A_l = \sigma_l^3 + 3\sigma_l^2 + 2\sigma_l - 6, \;\; B_l = 6(1+\sigma_l), \;\; C_l = \sigma_l(2\sigma_l^2 + 3\sigma_l - 5),$$

we consider the following variable mesh two-level implicit difference method of accuracy $O(k^2 + h_l^2)$ for the numerical solution of differential Eqs. (4.1) and (4.2):

$$\overline{U}_{l+1}^j - (1+\sigma_l)\overline{U}_l^j + \sigma_l \overline{U}_{l-1}^j = \frac{h_l^2}{6(1+\sigma_l)}[A_l(\overline{V}_{l+1}^j + \overline{U}_{t\,l+1}^j) + B_l(\overline{V}_l^j + \overline{U}_{t\,l}^j) + C_l(\overline{V}_{l-1}^j + \overline{U}_{t\,l-1}^j)] + O(k^2h_l^2 + h_l^4), \tag{53.1}$$

$$\begin{aligned}
&\overline{V}_{l+1}^j - (1+\sigma_l)\overline{V}_l^j + \sigma_l \overline{V}_{l-1}^j = \frac{h_l^2}{6(1+\sigma_l)}[A_l(\overline{F}_{l+1}^j + \overline{V}_{t\,l+1}^j) + B_l(\overline{F}_l^j + \overline{V}_{t\,l}^j) + C_l(\overline{F}_{l-1}^j + \overline{V}_{t\,l-1}^j)] + O(k^2h_l^2 + h_l^4), \\
&l = 1(1)N, \quad j = 0, 1, 2, \ldots
\end{aligned} \tag{53.2}$$

where $\overline{F}_l^j = f(x_l, \overline{t}_j, \overline{U}_l^j, \overline{V}_l^j, \overline{U}_{xl}^j, \overline{V}_{xl}^j)$ and the approximations $\overline{F}_{l\pm 1}^j$ are already defined in Section 2 with $\theta = 1/2$.

Also, a lower order variable mesh method of accuracy $O(k^2 + h_l^2)$ for the solution of differential equations (32.1) and (32.2) may be written as:

$$\widehat{U}_{l+1}^j - (1+\sigma_l)\widehat{U}_l^j + \sigma_l \widehat{U}_{l-1}^j = \frac{h_l^2}{6}[(\sigma_l - 1)\widehat{V}_{l+1}^j + (1+\sigma_l)(1+\sigma_l+\sigma_l^2)\widehat{V}_l^j - \sigma_l^2(\sigma_l - 1)\widehat{V}_{l-1}^j] + O(k^2h_l^3 + h_l^4), \tag{54.1}$$

$$[A_l^j + \frac{(\sigma_l - 1)}{3} h_l A_{x_l}^j][\widehat{V}_{l+1}^j - (1+\sigma_l)\widehat{V}_l^j + \sigma_l \widehat{V}_{l-1}^j]$$
$$= \frac{h_l^2}{6}[(\sigma_l - 1)(\widehat{G}_{l+1}^j - \widehat{U}_{tt_{l+1}}^j) + (1+\sigma_l)(1+\sigma_l+\sigma_l^2)(\widehat{G}_l^j - \widehat{U}_{tt_l}^j) - \sigma_l^2(\sigma_l - 1)(\widehat{G}_{l-1}^j - \widehat{U}_{tt_{l-1}}^j)]$$
$$+ O(k^2 h_l^2 + h_l^4), \quad \sigma_l \neq 1, \quad l = 1(1)N, \quad j = 1, 2, \ldots \tag{54.2}$$

where $\widehat{G}_l^j = g(x_l, t_j, \widehat{U}_l^j, \widehat{V}_l^j, \widehat{U}_{t_l}^j, \widehat{U}_{x_l}^j, \widehat{V}_{x_l}^j)$ and the approximations $\widehat{G}_{l\pm 1}^j$ are as defined in Section 3 for $0 < \tau < 1$. Also, note that the coefficients $A_l, B_l, C_l$ in (53.1) and (53.2) are positive if $\sigma_l > 1$.

For $\sigma_l = 1$ (uniform mesh case), that is, for $h_{l+1} = h_l = h$, method (53.1) and (53.2) for $\theta = 1/2$ and method (54.1) and (54.2) for $0 < \tau < 1$ reduces to the following two-level and three-level implicit difference methods of $O(k^2 + h^2)$ for the solution of differential Eqs. (4.1)–(4.2) and (32.1)–(32.2), respectively:

$$\delta_x^2 \overline{U}_l^j = h^2(\overline{V}_l^j + \overline{U}_{tl}^j) + O(k^2h^2 + kh^4 + h^4), \tag{55.1}$$

$$\delta_x^2 \overline{V}_l^j = h^2(\overline{F}_l^j + \overline{V}_{tl}^j) + O(k^2h^2 + kh^4 + h^4), \quad l = 1(1)N, \ j = 0, 1, 2, \ldots \tag{55.2}$$

$$\delta_x^2 \widehat{U}_l^j = h^2 \widehat{V}_l^j + O(k^2h^2 + h^4), \tag{56.1}$$

$$A_l^j \delta_x^2 \widehat{V}_l^j = h^2(\widehat{G}_l^j - \widehat{U}_{tt_l}^j) + O(k^2h^2 + h^4), \quad l = 1(1)N, \ j = 1, 2, \ldots \tag{56.2}$$

In this section, we conducted some numerical experiments to illustrate the methods (12.1)–(12.2) and (35.1)–(35.2) developed in previous sections for the solution of fourth-order differential Eqs. (1) and (26), respectively by solving some problems. We have compared our results with the results obtained by using the methods (53.1)–(53.2) and (54.1)–(54.2), respectively. The exact solutions and the region of integration are provided in each case. The right hand side functions, initial and boundary conditions are obtained using the exact solution as a test procedure. For computing the solution of (26), we have chosen $\tau = 0.5$. We have used iterative methods for solving coupled system of equations. If the system of equations is linear, we have used Gauss–Seidel method whereas in the non-linear case, the system is solved using the Newton's non-linear Gauss–Seidel method (see [24–26]). While using the Newton's method, the iterations were stopped when absolute error tolerance $\leq 10^{-12}$ was achieved. All computations are performed using MATLAB programming language.

Note that the difference method (35.1) and (35.2) for solving Eq. (26) is a three-level method. The values of $u$ and $v$ are known from initial conditions. To begin any computation, it is necessary to know the values of $u$ and $v$ of required accuracy at $t = k$. We discuss in this section an explicit method of $O(k^2)$ for $u$ and $v$ at the first time level, i.e. at $t = k$, which can be applied to problems in Cartesian and polar coordinates. Since the values of $u$ and $u_t$ are given explicitly at $t = 0$, so all their successive tangential partial derivatives are known at $t = 0$, i.e. the values of

$$\frac{\partial^s u_l^0}{\partial x^s}, \quad \frac{\partial^s v_l^0}{\partial x^s}, \quad \frac{\partial^{s+1} u_l^0}{\partial x^s \partial t}, \quad \frac{\partial^{s+1} v_l^0}{\partial x^s \partial t}, \quad s = 0, 1, \ldots$$

are known at $t = 0$. The prescribed differential Eq. (26) may be re-written as:

$$\frac{\partial^2 u}{\partial t^2} = -A(x,t,u,u_{xx})\frac{\partial^4 u}{\partial x^4} + g(x,t,u,v,u_t,u_x,v_x) \tag{57}$$

Differentiating (57) twice successively with respect to $x$, we get

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left[-A(x,t,u,u_{xx})\frac{\partial^4 u}{\partial x^4} + g(x,t,u,v,u_t,u_x,v_x)\right] \tag{58}$$

We consider the following explicit difference methods of $O(k^2)$ for $u$ and $v$ at $t = k$

$$U_l^1 = U_l^0 + kU_{t_l}^0 + \frac{k^2}{2}U_{tt_l}^0 + O(k^3) \tag{59}$$

$$V_l^1 = V_l^0 + kV_{t_l}^0 + \frac{k^2}{2}V_{tt_l}^0 + O(k^3) \tag{60}$$

Thus using the initial values and their successive tangential derivative values, we can obtain the values of $U_{tt_l}^0$ and $V_{tt_l}^0$ from (57) and (58), respectively. Then finally, using these values in (59) and (60), we can compute the values of $u$ and $v$ at $t = k$.

For uniform mesh, we take $\sigma_l = \sigma = 1$ and for variable mesh, the interval $[a, b]$ in $x$-direction is divided into $(N+1)$ parts with

$$a = x_0 < x_1 < \ldots < x_N < x_{N+1} = b, \quad \text{where} \quad h_l = x_l - x_{l-1}, \quad l = 1(1)N+1 \quad \text{and}$$
$$\sigma_l = (h_{l+1}/h_l) > 0, \quad l = 1(1)N.$$

**Table 1a**
Maximum absolute errors for Example 1 at $t = 1.0$ for uniform mesh.

| $N+1$ | | $O(k^2+kh^2+h^4)$-method (25.1) and (25.2) | | | $O(k^2+h^2)$-method (55.1) and (55.2) | | |
|---|---|---|---|---|---|---|---|
| | | $\alpha=0$ | $\alpha=10$ | $\alpha=20$ | $\alpha=0$ | $\alpha=10$ | $\alpha=20$ |
| 8 | $u$ | 4.8546(−05) | 2.2453(−05) | 1.2951(−05) | 1.0652(−02) | 6.9810(−03) | 5.6685(−03) |
| 16 | $u$ | 3.0106(−06) | 1.3881(−06) | 7.9713(−07) | 2.6369(−03) | 1.7429(−03) | 1.4184(−03) |
| 32 | $u$ | 1.8778(−07) | 8.6509(−08) | 4.9631(−08) | 6.5762(−04) | 4.3556(−04) | 3.5467(−04) |
| 64 | $u$ | 1.1626(−08) | 5.3712(−09) | 3.1145(−09) | 1.6430(−04) | 1.0888(−04) | 8.8671(−05) |

**Table 1b**
Maximum absolute errors for Example 1 at $t = 1.0,\ \sigma = 1.04$ for variable mesh.

| $N+1$ | | $O(k^2+kh_l+h_l^3)$-method (12.1) and (12.2) | | | $O(k^2+h_l^2)$-method (53.1) and (53.2) | | |
|---|---|---|---|---|---|---|---|
| | | $\alpha=0$ | $\alpha=10$ | $\alpha=20$ | $\alpha=0$ | $\alpha=10$ | $\alpha=20$ |
| 8 | $u$ | 7.0744(−05) | 4.1221(−05) | 3.1869(−05) | 7.0345(−03) | 4.6117(−03) | 3.7403(−03) |
| 16 | $u$ | 8.5931(−06) | 5.7542(−06) | 4.7214(−06) | 1.8600(−03) | 1.2354(−03) | 1.0081(−03) |
| 32 | $u$ | 1.7925(−06) | 1.2665(−06) | 1.0823(−06) | 5.7255(−04) | 3.8168(−04) | 3.1200(−04) |
| 64 | $u$ | 7.0275(−07) | 5.0342(−07) | 4.3396(−07) | 2.5900(−04) | 1.7370(−04) | 1.4280(−04) |

Since

$$b = x_{N+1} - x_0 = (x_{N+1} - x_N) + (x_N - x_{N-1}) + \cdots + (x_1 - x_0) = h_{N+1} + h_N + \cdots + h_1$$
$$= h_1(1 + \sigma_1 + \sigma_1\sigma_2 + \cdots + \sigma_1\sigma_2\cdots\sigma_N),$$

so we obtain the first mesh spacing in $x$-direction as

$$h_1 = \frac{b}{1 + \sigma_1 + \sigma_1\sigma_2 + \cdots + \sigma_1\sigma_2\cdots\sigma_N} \tag{61}$$

For simplicity, we may consider $\sigma_l = \sigma$ (a constant $\neq 1$), $l = 1(1)N$, then $h_1$ reduces to

$$h_1 = \frac{b(1-\sigma)}{1-\sigma^{N+1}}, \quad \sigma \neq 1. \tag{62}$$

Thus, by prescribing the value of $N$ and $\sigma$, we can calculate $h_1$ from the above relation and the remaining mesh lengths in $x$-direction are known by $h_{l+1} = \sigma h_l,\ l = 1(1)N$. Throughout our computation we have used the time step $k = 1.6/(N+1)^2$ for finding the solution at $t = 1$ and $k = 0.8/(N+1)^2$ at $t = 2$, for variable mesh. The difference method (25.1) and (25.2) for the solution of differential Eq. (1) is fourth-order accurate in space and second-order accurate in time in case of uniform mesh, that is, for $\sigma_l = 1$, $h_{l+1} = h_l = h$. For computation, we choose the mesh ratio parameter in case of uniform mesh as $\lambda = k/h^2 = 1.6$ for Eq. (1). Since $k \propto h^2$, this implies the method (25.1) and (25.2) is $O(h^4)$ accurate in space.

**Example 1.**

$$\frac{\partial^4 u}{\partial x^4} - 2\frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial^2 u}{\partial t^2} = \alpha u(u_{xx} - u_t) + f(x,t), \quad 0 < x < 1,\ t > 0 \quad \text{(non-linear equation)} \tag{63}$$

The exact solution is $u = e^{-t}\sin(\pi x)$. The maximum absolute errors for uniform mesh (for $\sigma_l = 1$) and for variable mesh for a fixed $\sigma = 1.04$ are tabulated in Table 1a and Table 1b, respectively at $t = 1.0$ for various values of $\alpha$. The 3D graphs of numerical solution using method (12.1) and (12.2) vs exact solution for variable mesh are plotted in Fig. 1 at $t = 1$ for $\alpha = 20$.

**Example 2.** For the determination of the transverse deflection of points of the rod from their position of equilibrium, we solve the following physical problem considered in [16]:

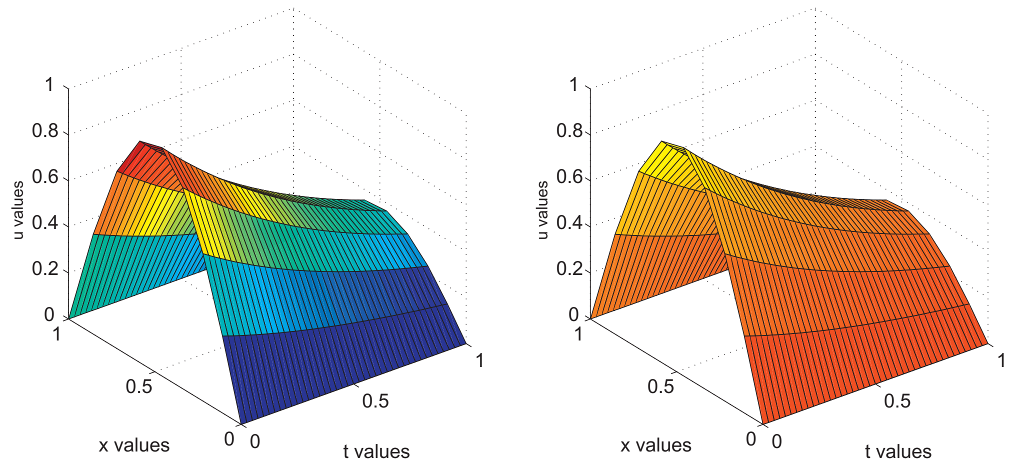
$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} + \eta u = f(x,t), \quad 0 < x < 1,\ t > 0 \tag{64}$$

where $\eta$ is the ratio of coefficient of friction to the linear mass density of the string. The exact solution is $u = \sin\pi x\cos t$. The maximum absolute errors are tabulated in Table 2 at $t = 1.0,\ 2.0$ for a fixed $\sigma = 0.96$ and for various values of $\eta$.

**Example 3.**

$$\left(1 + u^2 + u_{xx}^2\right)\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = \alpha u(u_x - u_{xx}) + f(x,t), \quad 0 < x < 1, \quad t > 0 \quad \text{(quasi-linear equation)} \tag{65}$$

The exact solution is $u = x^4\sinh t$. The maximum absolute errors are tabulated in Table 3 at $t = 1.0$ for a fixed $\sigma = 1.06$ and various values of $\alpha$.




(a) Numerical Solution (b) Exact Solution

**Fig. 1.** Example 1: The graph of numerical and exact solution at $t = 1$, $\sigma = 1.04$, $N + 1 = 8$ and $\alpha = 20$.

**Table 2**
Maximum absolute errors for Example 2 at $t = 1.0$ and $t = 2.0$, $\sigma = 0.96$.

| $N+1$ | | $O(k^2 + h_l^3)$-method (35.1) and (35.2) | | | | | |
|---|---|---|---|---|---|---|---|
| | | $t = 1.0$ | | | $t = 2.0$ | | |
| | | $\eta = 0.1$ | $\eta = 0.01$ | $\eta = 0.001$ | $\eta = 0.1$ | $\eta = 0.01$ | $\eta = 0.001$ |
| 8 | $u$ | 8.5443(−04) | 8.5529(−04) | 8.5537(−04) | 7.2379(−04) | 7.2657(−04) | 7.2685(−04) |
| | $u_{xx}$ | 9.8688(−03) | 9.8764(−03) | 9.8771(−03) | 6.6571(−03) | 6.6817(−03) | 6.6841(−03) |
| 16 | $u$ | 6.5713(−05) | 6.5828(−05) | 6.5840(−05) | 4.9613(−05) | 4.9950(−05) | 4.9983(−05) |
| | $u_{xx}$ | 7.4238(−04) | 7.4350(−04) | 7.4361(−04) | 5.7302(−04) | 5.7199(−04) | 5.7189(−04) |
| 32 | $u$ | 9.0902(−06) | 9.1083(−06) | 9.1102(−06) | 6.2920(−06) | 6.3395(−06) | 6.3442(−06) |
| | $u_{xx}$ | 1.4775(−04) | 1.4785(−04) | 1.4786(−04) | 1.1837(−04) | 1.1864(−04) | 1.1867(−04) |
| 64 | $u$ | 3.0067(−06) | 3.0128(−06) | 3.0134(−06) | 2.1333(−06) | 2.1491(−06) | 2.1507(−06) |
| | $u_{xx}$ | 5.4602(−05) | 5.4648(−05) | 5.4653(−05) | 6.0983(−05) | 6.0934(−05) | 6.0929(−05) |

**Table 3**
Maximum absolute errors for Example 3 at $t = 1.0$, $\sigma = 1.06$.

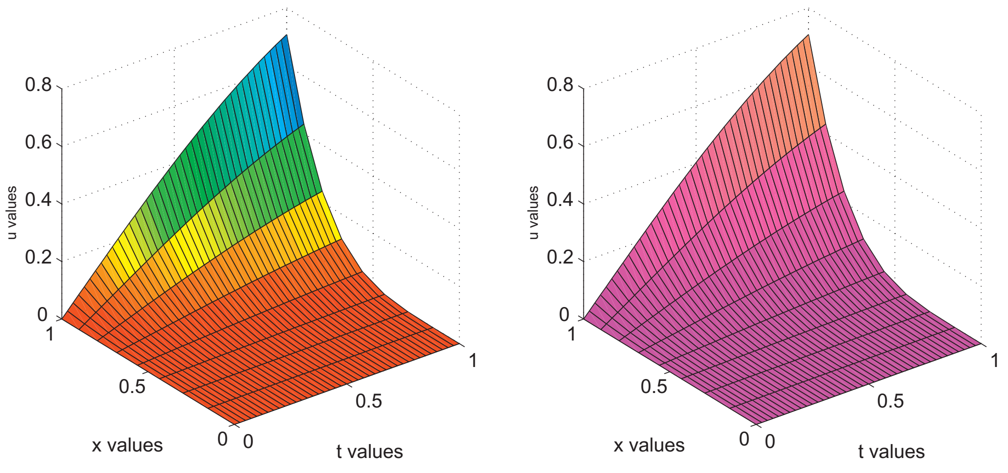
| $N+1$ | | $O(k^2 + h_l^3)$-method (35.1) and (35.2) | | | $O(k^2 + h_l^2)$-method (54.1) and (54.2) | | |
|---|---|---|---|---|---|---|---|
| | | $\alpha = 10$ | $\alpha = 20$ | $\alpha = 40$ | $\alpha = 10$ | $\alpha = 20$ | $\alpha = 40$ |
| 8 | $u$ | 2.5863(−03) | 2.5317(−03) | 2.4305(−03) | 4.9765(−03) | 4.9289(−03) | 4.8418(−03) |
| | $u_{xx}$ | 2.6687(−02) | 2.6130(−02) | 2.5096(−02) | 1.8970(−03) | 1.3675(−03) | 9.8172(−04) |
| 16 | $u$ | 1.2182(−04) | 1.1940(−04) | 1.1493(−04) | 1.4311(−03) | 1.4212(−03) | 1.3990(−03) |
| | $u_{xx}$ | 1.2335(−03) | 1.2089(−03) | 1.1634(−03) | 4.4764(−04) | 3.3492(−04) | 2.0408(−04) |
| 32 | $u$ | 7.2096(−06) | 7.0619(−06) | 6.7972(−06) | 5.4395(−04) | 5.4121(−04) | 5.3466(−04) |
| | $u_{xx}$ | 7.4656(−05) | 7.3322(−05) | 7.0922(−05) | 1.2241(−04) | 9.5291(−05) | 5.8963(−05) |
| 64 | $u$ | 1.0633(−06) | 1.0390(−06) | 9.9914(−07) | 3.4552(−04) | 3.4417(−04) | 3.4077(−04) |
| | $u_{xx}$ | 1.2351(−05) | 1.2186(−05) | 1.1918(−05) | 6.4236(−05) | 4.8981(−05) | 3.7246(−05) |

**Example 4.** To solve numerically Eq. (49) whose exact solution is $u = r^4 \sin r \sin t$. The maximum absolute errors are tabulated in Table 4 at $t = 1.0,\ 2.0$ for a fixed $\sigma = 0.94$ and for $\alpha = 1, 2$. The 3D graphs of numerical solution vs exact solution are plotted in Fig. 2 at $t = 1$ for $\alpha = 1$.

**Example 5.** Consider the second-order Benjamin–Ono equation:

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u^2}{\partial x^2} = 0, \quad -25 < x < 25,\ t > 0 \tag{66}$$

This kind of equation is one of the most significant non-linear PDEs, which is used to describe the long waves in shallow water and also in the study of many other physical phenomenon, such as the percolation of water in the porous surface of a horizontal layer of material. Fu et al. [27] constructed the exact periodic solutions of Eq. (66) using the Jacobi elliptic function expansion




(a) Numerical Solution (b) Exact Solution

**Fig. 2.** Example 4: The graph of numerical and exact solution at $t = 1$, $\sigma = 0.94$, $N + 1 = 8$ and $\alpha = 1$.

**Table 4**
Maximum absolute errors for Example 4 at $t = 1.0$ and $t = 2.0$, $\sigma = 0.94$

| $N+1$ | | $O(k^2 + h_l^3)$-method (35.1) and (35.2) | | | |
|---|---|---|---|---|---|
| | | $t = 1.0$ | | $t = 2.0$ | |
| | | $\alpha = 1$ | $\alpha = 2$ | $\alpha = 1$ | $\alpha = 2$ |
| 8 | $u$ | 3.9009(−04) | 2.3296(−03) | 4.2917(−04) | 2.5325(−03) |
| | $u_{xx}$ | 1.5827(−02) | 8.0582(−02) | 1.7517(−02) | 8.6534(−02) |
| 16 | $u$ | 5.3989(−05) | 3.4962(−04) | 6.1110(−05) | 3.7713(−04) |
| | $u_{xx}$ | 3.6585(−03) | 2.1261(−02) | 3.8984(−03) | 2.2963(−02) |
| 32 | $u$ | 1.3226(−05) | 9.0634(−05) | 1.4930(−05) | 9.7630(−05) |
| | $u_{xx}$ | 1.3507(−03) | 8.4832(−03) | 1.4399(−03) | 9.1583(−03) |
| 64 | $u$ | 7.2966(−06) | 5.1297(−05) | 8.2431(−06) | 5.5299(−05) |
| | $u_{xx}$ | 9.0550(−04) | 5.7976(−03) | 9.6465(−04) | 6.2615(−03) |

**Table 5**
Maximum absolute errors for Example 5 at various time levels for a uniform mesh with $h = 0.1,\ k = 0.01$.

| $t$ | $O(k^2 + h^4)$-method (44.1) and (44.2) | | Method discussed in [28] |
|---|---|---|---|
| | $u$ | $u_{xx}$ | $u$ |
| 5 | 3.2357(−07) | 1.9753(−07) | 3.3988(−04) |
| 10 | 1.3715(−06) | 6.2901(−07) | 5.2273(−04) |
| 15 | 4.1498(−06) | 1.8354(−06) | 8.2328(−04) |
| 20 | 1.0743(−05) | 4.7400(−06) | 1.2596(−03) |

method having the form

$$u(x,t) = \frac{1}{2}b^2 - 4l^2 + 6l^2\tanh^2[l(x - bt)]. \tag{67}$$

By using the difference method (44.1) and (44.2) for uniform mesh, we demonstrate the above traveling wave solution (67) of Eq. (66) through numerical computation and compare our results with the lattice Boltzmann method proposed in [28]. We first take the same physical constants as in [28]: $l = 0.3,\ b = 0.01$ and the computation domain is taken as $[-25, 25]$ with $h = 0.1,\ k = 0.01$. The maximum absolute errors (for $\sigma_l = 1$) are tabulated in Table 5 at various time levels $t = 5, 10, 15$ and 20. The 2D graph of numerical solution vs analytical solution is plotted in Fig. 3 at $t = 5$.

**Example 6.** Consider the good Boussinesq equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u^2}{\partial x^2} - \frac{\partial^4 u}{\partial x^4}, \quad 0 < x < 1, \quad t > 0 \tag{68}$$

The above equation which belongs to the KdV family of equations describes the propagation of long waves in shallow water [29] and occurs in various other physical applications such as non-linear lattice waves, ion sound waves in plasma and many more

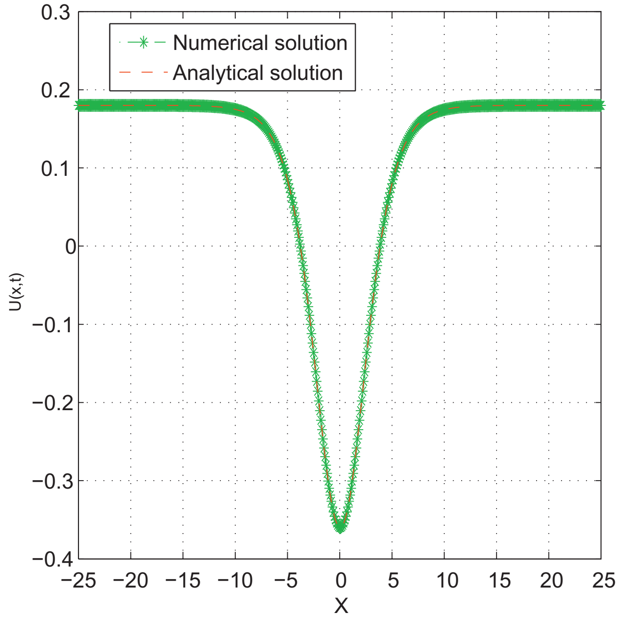



**Fig. 3.** Example 5: Comparison between numerical and analytical solution for Eq. (66) at $t = 5$ for $h = 0.1$ and $k = 0.01$.

**Table 6a**
Maximum absolute errors for Example 6 at various time levels for a uniform mesh with $k = 0.05$.

| $t$ | Parameters | | $O(k^2 + h^4)$-method (44.1) and (44.2) | | Quintic B-spline method discussed in [31] |
|---|---|---|---|---|---|
| | | | $u$ | $u_{xx}$ | $u$ |
| 0.5 | $h = 1/40$ | $x^0 = 30$ | 7.8998(−07) | 4.2242(−06) | 8.2943(−07) |
| 1.0 | $h = 1/60$ | $x^0 = 40$ | 7.5071(−09) | 3.7560(−08) | 7.3326(−09) |
| 1.5 | $h = 1/80$ | $x^0 = 50$ | 5.7588(−11) | 2.2115(−09) | 6.4525(−11) |
| 2.0 | $h = 1/100$ | $x^0 = 60$ | 2.9068(−13) | 3.2831(−13) | 5.2066(−13) |

(see [30]). Siddiqi and Arshed [31] used the quintic B-spline collocation method to determine the numerical solution of (68) subject to the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0$$
$$u_{xx}(0,t) = 0, \quad u_{xx}(1,t) = 0, \quad t > 0$$

The exact solution of (68) according to [31] is

$$u(x,t) = -A\,\mathrm{sech}^2\left[\sqrt{\frac{A}{6}}(x - ct + x^0)\right] - \left(b + \frac{1}{2}\right). \tag{69}$$

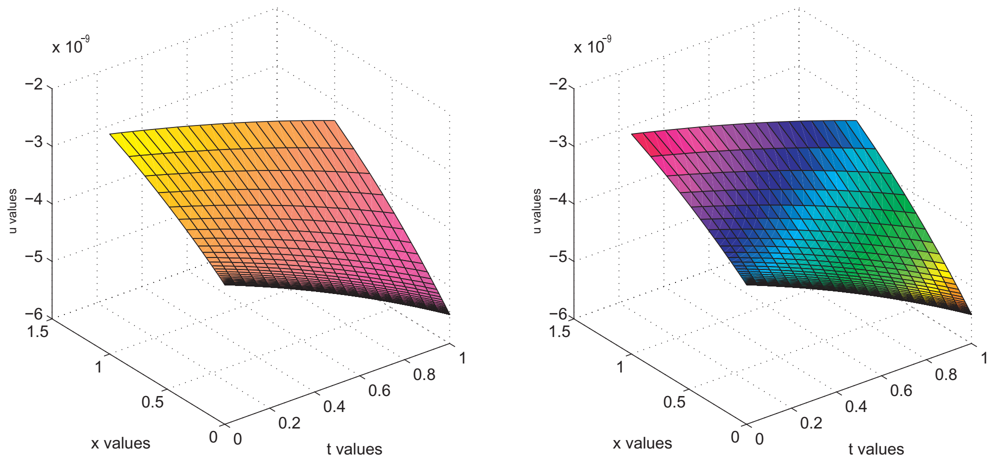
Here $c$ is the velocity, $A$ is amplitude of the pulse, $b$ is an arbitrary parameter and $x^0$ is the initial position. We apply the difference method (44.1) and (44.2) for uniform mesh to determine the numerical solution of Eq. (68) and compare the results obtained with quintic B-spline method of [31]. Excellent agreement is noted. We take the same theoretical parameters as in [31]: $A = 0.369,\ b = -\frac{1}{2}$ and $c = 0.868$. The maximum absolute errors (for $\sigma_l = 1$) are tabulated in Table 6a at various time levels $t = 0.5, 1.0, 1.5$ and 2.0. Also, we have solved Eq. (68) numerically using difference method (35.1) and (35.2) for variable mesh with the boundary conditions corresponding to the data from the analytical solution (69) and compared the maximum absolute errors obtained using difference method (54.1) and (54.2) for a fixed $\sigma = 1.2$ in Table 6b. The 3D graphs of numerical solution using method (35.1) and (35.2) vs exact solution are plotted in Fig. 4 at $t = 1$.

## 6. Concluding remarks

In this paper, using coupled approach we have presented a kind of new and accurate implicit two-level and three-level variable mesh finite difference methods of $O(k^2 + kh_l + h_l^3)$ and $O(k^2 + h_l^3)$ accuracy based on Numerov type discretization for the solution of special types of fourth-order non-linear time dependent parabolic partial differential Eqs. (1) and (26), respectively using three spatial grid points. Numerical results indicate that the proposed methods are computationally more efficient than

**Table 6b**
Maximum absolute errors for Example 6 at various time levels for a variable mesh for $\sigma = 1.2$, $k = 0.05$.

| $t$ | Parameters | | $O(k^2+h_l^3)$-method (35.1) and (35.2) | | $O(k^2+h_l^2)$-method (54.1) and (54.2) | |
|---|---|---|---|---|---|---|
| | $N$+1 | | $u$ | $u_{xx}$ | $u$ | $u_{xx}$ |
| 0.5 | 40 | $x^0 = 30$ | 1.6799(−14) | 5.6869(−13) | 3.2052(−12) | 4.4789(−11) |
| 1.0 | 60 | $x^0 = 40$ | 1.3098(−16) | 7.4029(−15) | 8.0684(−15) | 8.9990(−13) |
| 1.5 | 80 | $x^0 = 50$ | 1.1890(−18) | 1.2935(−16) | 1.3192(−16) | 2.1428(−14) |
| 2.0 | 100 | $x^0 = 60$ | 9.5574(−21) | 3.1718(−18) | 3.6099(−18) | 5.1623(−16) |




(a) Numerical Solution (b) Exact Solution

**Fig. 4.** Example 6: The graph of numerical and exact solution at $t = 1$, $\sigma = 1.2$, $N + 1 = 40$, $k = 0.05$ and $x^0 = 30$.

the corresponding variable mesh methods of $O(k^2 + h_l^2)$ accuracy. The order of convergence may be obtained for uniform mesh by using the formula $\log(e_{h_1}/e_{h_2})/\log(h_1/h_2)$, where $e_{h_1}$ and $e_{h_2}$ are maximum absolute errors for two mesh widths $h_1$ and $h_2$, respectively. Numerical results from Table 1 a confirm that the difference method (25.1) and (25.2) in case of uniform mesh for the solution of differential Eq. (1) is fourth-order accurate, when grid size in time-direction is directly proportional to the square of grid size in space-direction. Hence, for $\sigma_l = 1$, the proposed difference methods reduces to fourth-order accurate uniform mesh methods. There was no need to discretize the boundary conditions and without using fictitious points, we obtained the solution, thereby eliminating the typical difficulty encountered in using conventional central difference methods. Also we obtained the numerical solution of $u_{xx}$ as a by-product of method (35.1) and (35.2), which is quite often of interest in many physical problems. It is shown that the proposed two-level finite difference method (25.1) and (25.2) is unconditionally stable when applied to a linear equation. The proposed three-level numerical method is also applicable to singular equation which produces oscillation free solutions even in the vicinity of the singularity, which is an advantage of our work. Numerical results show that the derived methods are capable of solving effectively a large number of non-linear equations like the second-order Benjamin–Ono equation and the good Boussinesq equation with high accuracy. The developed methods are straightforward, concise and effective in solving a large variety of non-linear PDEs. The proposed methods may be extended to solve 2D and 3D fourth-order non-linear time dependent parabolic PDEs. Further research will be centered on the applications of new methods to some more practical science and engineering problems such as vibrations of a homogenous beam, inelastic flows and various variants of the Boussinesq equation.

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