Research Article

Ranjan Kumar Mohanty* and Deepti Kaur

High Accuracy Compact Operator Methods for Two-Dimensional Fourth Order Nonlinear Parabolic Partial Differential Equations

DOI: 10.1515/cmam-2016-0047
Received August 16, 2016; revised December 28, 2016; accepted January 1, 2017

Abstract: In this study, we develop and implement numerical schemes to solve classes of two-dimensional fourth-order partial differential equations. These methods are fourth-order accurate in space and second-order accurate in time and require only nine spatial grid points of a single compact cell. The proposed discretizations allow the use of Dirichlet boundary conditions only without the need to discretize the derivative boundary conditions and thus avoids the use of ghost points. No transformation or linearization technique is used to handle nonlinearity and the obtained block tri-diagonal nonlinear system has been solved by Newton's block iteration method. It is discussed how our formulation is able to tackle linear singular problems and it is ensured that the methods retain their orders and accuracy everywhere in the solution region. The proposed two-level method is shown to be unconditionally stable for a class of two-dimensional fourth-order linear parabolic equation. We also discuss the alternating direction implicit (ADI) method for solving two-dimensional fourth-order linear parabolic equation. The proposed difference methods has been successfully tested on the two-dimensional vibration problem, Boussinesq equation, extended Fisher–Kolmogorov equation and Kuramoto–Sivashinsky equation. Numerical results demonstrate that the schemes are highly accurate in solving a large class of physical problems.

Keywords: Transverse Vibrations, Compact Operator Method, Extended Fisher–Kolmogorov Equation, Kuramoto–Sivashinsky Equation, Block Tri-Diagonal System, ADI Method

MSC 2010: 65M06, 65M12, 65M22

1 Introduction

We consider the following two-dimensional fourth-order parabolic partial differential equation (PDE):

$$A(x,y,t,u,\nabla^2 u)\nabla^4 u + \frac{\partial^2 u}{\partial t^2} = f(x,y,t,u,u_t,u_x,u_y,\nabla^2 u,\nabla^2 u_x,\nabla^2 u_y), \quad (x,y)\in\Omega,\ t>0, \tag{1}$$

where $\Omega = \{(x,y) : a < x, y < b\}$ is a bounded domain of $\mathbb{R}^2$, $\nabla^2 u(x,y,t) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ represents the two-space-dimensional Laplacian of the function $u(x,y,t)$ and

$$\nabla^4 \equiv \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

is the two-dimensional biharmonic operator. We assume that the solution $u(x,y,t)$ is smooth enough to maintain the order and accuracy of the difference schemes under consideration. The initial conditions associated

***Corresponding author: Ranjan Kumar Mohanty:** Department of Applied Mathematics, Faculty of Mathematics and Computer Science, South Asian University, Akbar Bhawan, Chanakyapuri, New Delhi 110021, India, e-mail: rmohanty@sau.ac.in
Deepti Kaur: Department of Mathematics, Faculty of Mathematical Sciences, University of Delhi, Delhi 110007, India, e-mail: deeptimaths15@gmail.com

with (1) are of the form

$$u(x,y,0)=u_0(x,y),\quad (x,y)\in\Omega, \tag{2.1}$$

$$\frac{\partial u}{\partial t}(x,y,0)=u_1(x,y),\quad (x,y)\in\Omega, \tag{2.2}$$

and boundary conditions are given by

$$u(x,a,t)=a_{10}(x,t),\quad \frac{\partial^2 u}{\partial y^2}(x,a,t)=a_{20}(x,t),\quad a\le x\le b,\ t>0, \tag{3.1}$$

$$u(x,b,t)=a_{11}(x,t),\quad \frac{\partial^2 u}{\partial y^2}(x,b,t)=a_{21}(x,t),\quad a\le x\le b,\ t>0, \tag{3.2}$$

$$u(a,y,t)=b_{10}(y,t),\quad \frac{\partial^2 u}{\partial x^2}(a,y,t)=b_{02}(y,t),\quad a\le y\le b,\ t>0, \tag{3.3}$$

$$u(b,y,t)=b_{11}(y,t),\quad \frac{\partial^2 u}{\partial x^2}(b,y,t)=b_{12}(y,t),\quad a\le y\le b,\ t>0, \tag{3.4}$$

where $u_0, u_1, a_{10}, a_{20}, a_{11}, a_{21}, b_{10}, b_{02}, b_{11}$ and b_{12} are sufficiently differentiable functions. We refer to the PDE (1) together with the initial and boundary conditions (2.1)–(2.2) and (3.1)–(3.4) as problem (P1). Equation (1) occurs in several interesting problems, classical examples can be found in thin beams and plates, strain gradient elasticity, in the propagation of long waves in shallow water, in plate vibration and seismological problems, etc. (see [29, 32]).

The classical theory of transverse vibrations of a square homogenous thin plate with each side of length L leads to the following fourth-order PDE in two space variables:

$$\sigma\nabla^4 u+\mu\frac{\partial^2 u}{\partial t^2}=0,\quad 0<x,y<L,\ t>0,$$

where $u(x,y,t)$ is the vertical deflection at location (x,y) at time t, $\mu>0$ is the mass density per unit area and $\sigma>0$ is the flexural rigidity. This theory is an extension of Euler–Bernoulli beam theory and was developed using the assumptions proposed by Kirchhoff [29] to model the vibrations of a thin plate subject to pure bending with small deflection. It is an important mathematical model used in structural mechanics and seismology.

The two-space-dimensional Boussinesq equation is of the form

$$\frac{\partial^2 u}{\partial t^2}=\nabla^2 u-\alpha\nabla^2 u^2-\beta\nabla^4 u,\quad (x,y)\in\Omega,\ t>0,$$

where $u(x,y,t)$ is the surface elevation, $\beta>0$ is the dispersion coefficient and α is an amplitude parameter is also considered. This equation is introduced to describe the propagation of gravity waves on the surface of water [32] and has many applications; e.g., for ion-acoustic wave in plasma, magnetohydrodynamics wave in plasma, longitudinal dispersive wave in elastic rods and pressure wave in liquid-gas bubble mixtures and so on. This equation belongs to the KdV family of equations and was first introduced by Boussinesq [3] in 1872.

Another class of two-dimensional time-dependent biharmonic equation studied in the paper is of the form

$$B(x,y,t,u,\nabla^2 u)\nabla^4 u+\frac{\partial u}{\partial t}=g(x,y,t,u,u_x,u_y,\nabla^2 u,\nabla^2 u_x,\nabla^2 u_y),\quad (x,y)\in\Omega,\ t>0, \tag{4}$$

subject to the initial condition (2.1) and boundary conditions (3.1)–(3.4). The PDE (4) together with the initial and boundary conditions (2.1) and (3.1)–(3.4), respectively, is referred to as problem (P2). Equations of this form are fundamental parts of many applied mathematical models as discussed below.

The two-dimensional extended Fisher–Kolmogorov (EFK) equation can be written as

$$\frac{\partial u}{\partial t}+\gamma\nabla^4 u-\nabla^2 u+f(u)=0,\quad (x,y)\in\Omega,\ t>0, \tag{5}$$

where $f(u)=u^3-u$ and γ is a positive constant. When $\gamma=0$ in (5), the standard Fisher–Kolmogorov equation [2] is obtained. Dee and Van Saarlos [8] proposed (5) by adding a stabilizing fourth-order derivative term to the Fisher–Kolmogorov equation and called the model described by (5) as the EFK equation. Equation (5) arises in a range of applications such as propagation of domain walls in liquid crystals [34] and traveling waves in reaction-diffusion systems [2].

The two-dimensional Kuramoto–Sivashinsky (KS) equation in the space derivative form is expressed as

$$\frac{\partial u}{\partial t} + \nabla^4 u + \nabla^2 u + u\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) = 0, \quad (x, y) \in \Omega,\ t > 0. \tag{6}$$

The KS equation (6) has been derived for the purpose of explaining the intrinsic instabilities in laminar flame fronts and phase dynamics in reaction-diffusion systems [17]. This equation stands as a classical model for chaotic spatially extended systems and can be used to study the relations between chaotic dynamics at small scales and apparent stochastic behavior at large scales.

Another particular class of fourth-order nonlinear parabolic PDE considered in this study is of the form

$$\nabla^4 u - 2\left(\frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial^3 u}{\partial y^2 \partial t}\right) + \frac{\partial^2 u}{\partial t^2} = \psi(x, y, t, u), \quad (x, y) \in \Omega,\ t > 0. \tag{7}$$

We refer to the PDE (7) together with the initial conditions (2.1)–(2.2) and boundary conditions (3.1)–(3.4) as problem (P3).

Nonlinear fourth-order PDEs are useful in describing various physical phenomena and dynamic processes in science and engineering. Analytical solutions of these equations are usually not available. Since only limited classes of these equations can be solved by analytical means, numerical solution of these nonlinear PDEs are of practical importance. The numerical solution of these equations have attracted a lot of attention which has resulted in various finite difference, finite element and adomian decomposition methods. Conte [4] presented a finite difference approximation to the vibration problems in two space variables resulting in a quidiagonal system of equations (at most five non-zero elements in any one row). Fairweather and Gourlay [9] discussed some ADI methods for the solution of fourth-order two-space-variable problem arising in the study of transverse vibrations of a uniform plate. Andrade and McKee [1] derived high accuracy ADI methods for solving fourth-order parabolic equations with variable coefficients in one, two and three space dimensions. Twizell and Khaliq [30] developed a numerical method for the solution of two-dimensional fourth-order parabolic PDEs. Wazwaz [31] approached the variable coefficient fourth-order parabolic PDE in two space variables analytically to obtain the exact solution. Witelski and Bowen [33] constructed ADI schemes for the solution of two-dimensional higher order linear and nonlinear diffusion equations, particularly the fourth-order thin film equation for surface tension driven fluid flows. Danumjaya and Pani [6] analyzed the convergence of the approximate solution of the EFK equation using the second-order splitting combined with the orthogonal cubic spline collocation method. A finite element Galerkin method and optimal error estimates are derived for solving the two-dimensional EFK equation (5) in [7]. Gupta and Gupta [12] applied the homotopy perturbation transform method for solving singular fourth-order parabolic PDEs with variable coefficients. Liu, Li, Fang, He and Wang [18] studied a new coupling method based on new expanded mixed finite element scheme and finite element scheme for the two-dimensional fourth-order PDE of parabolic type. Khiari and Omrani [16] derived Crank–Nicolson-type finite difference method of $O(k^2 + h^2)$ in the L_∞-norm for the EFK equation in two space dimensions with Dirichlet boundary conditions, where $k > 0$ and $h > 0$ are the step sizes in the temporal and spatial dimensions, respectively. Recently, Kalogirou, Keaveny and Papageorgiou [14] presented an extensive numerical study of the two-dimensional KS equation. Further, Mohanty and Kaur [24–26] have derived high accuracy numerical methods based on Numerov-type and off-step discretization for the solution of one-dimensional fourth-order quasi-linear PDEs on a variable mesh by splitting the fourth-order equation into a coupled system of two second-order equations using only three spatial grid points which are capable of solving effectively a large number of nonlinear equations like the one-dimensional second-order Benjamin–Ono equation, the good Boussinesq equation, the KS equation and the EFK equation with high accuracy.

The emergence and growing recognition of *compact schemes* brought a renewed interest in the discretization of biharmonic problems. In this direction, Mohanty and Pandey [27] reported two sets of finite difference methods of order two and four over a rectangular domain for the numerical integration of the system of two-dimensional nonlinear biharmonic problems of second kind. In recent years, many researchers have studied some numerical methods for fourth-order elliptic equations [20, 22]. A combination of values of $u(x, t)$ and its derivative $u_x(x, t)$ are used to develop high order implicit difference formulas for fourth-order quasi-linear parabolic PDEs in one space variable with clamped BCs, i.e. when u and u_x is prescribed at the boundary in [19, 23]. Mohanty, Dai and Han [21] proposed a new compact fourth-order accurate difference method for solving the two-dimensional fourth-order elliptic boundary value problem with third order non-linear derivative terms subject to Dirichlet and Neumann boundary conditions. The numerical solution of time-dependent fourth-order problems (P1), (P2) and (P3) poses several challenges. Time stepping is perhaps critical issue for fourth-order parabolic equations. Stability requirement restrict time steps for explicit methods to be of $O(h^4)$ which is unreasonable hence implicit methods are essential. Also, the stencil size plays a key role in computation speed. Since far fewer operations are needed to discretize the Laplacian than discretizing the biharmonic, it is necessary to keep the domain of computation compact in order to make the computation efficient and reliable. It is also difficult to develop splitting schemes for two-dimensional problems as the spatial operator necessarily includes mixed derivative terms.

In the present article, we propose new implicit compact discretizations for the solution of two-dimensional fourth-order quasi-linear PDEs using splitting technique. These methods are second-order accurate in time and fourth-order accurate in space and involves only nine grid points of a single compact cell at each time level. The foremost advantage of developing these new methods is that no fictitious points for incorporating the boundary conditions are needed. The given boundary conditions are exactly satisfied and no approximations for derivatives need to be carried out at the boundary of the domain.

The layout of this article is as follows. In Section 2, we formulate and derive new three-level implicit compact operator method of accuracy two in time and four in space for the solution of two-dimensional quasi-linear problem (P1) using coupled approach. We mainly discretize the spatial derivatives u_x, u_y, $\nabla^2 u_x$, $\nabla^2 u_y$ using higher order approximations and the temporal derivatives u_t, u_{tt} using second-order approximations. In Section 3, an implicit two-level operator method is proposed for solving two-dimensional time-dependent biharmonic problem (P2) on a nine point compact stencil using the values of solution $u(x, y, t)$ and its spatial Laplacian as the unknowns. In Section 4, we discuss the application of the suggested methods to two-dimensional fourth-order equations with singular coefficients, wherein we modify our methods in such a way that the solution retains its order and accuracy everywhere in the vicinity of the singularity. In Section 5, linear stability of the proposed two-level compact operator schemes for problem (P3) is studied using matrix stability analysis and found to be unconditionally stable. Also, to ease the computation, we develop the corresponding ADI schemes for (P3) in Section 5. Finally, in Section 6, the results of a number of numerical experiments on problems of physical significance are given in order to demonstrate the accuracy and usefulness of the proposed methods. Concluding remarks about the present work are given at the end in Section 7.

2 Three-Level Compact Operator Method for (P1)

In this section, we derive a discrete approximation to problem (P1) which is fourth-order accurate in the spatial variables.

We discretize the domain Ω by placing a uniform mesh of spacing $h > 0$ in x and y directions with $(N+1)h = (b-a)$ such that $a = x_0 < x_1 < \cdots < x_N < x_{N+1} = b$ and $a = y_0 < y_1 < \cdots < y_N < y_{N+1} = b$, N being a positive integer. Let k be the step size in the time direction. Each grid point is denoted by (x_l, y_m, t_j), where

$$\begin{aligned} x_l &= a + lh, & l &= 0, 1, \ldots, N+1, \\ y_m &= a + mh, & m &= 0, 1, \ldots, N+1, \\ t_j &= jk, & j &= 0, 1, \ldots, J, \end{aligned}$$

with $Jk = T$, T is the final time of computation. The mesh ratio parameter is given by $\lambda = k/h^2$.

Note that the boundary conditions are given by (3.1)–(3.4). Since the grid lines are parallel to the coordinate axes and the value of u is exactly known on the boundary, this implies the values of its successive partial derivatives are also known on the boundary. For instance, on the plane $y = a$, u and u_{yy} are given, hence the values of u_x, u_{xx}, etc. are known there. This implies that the values of u and $\nabla^2 u = u_{xx} + u_{yy}$ are known at $y = a$. Similarly, u and $\nabla^2 u$ are known everywhere on the boundary of the solution domain.

For simplicity, we first consider $A = A(x, y, t)$. To motivate the present study, we split equation (1) by setting $v = \nabla^2 u$ to obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = v, \qquad (x, y) \in \Omega, \tag{8.1}$$

$$A(x, y, t)\Big(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\Big) + \frac{\partial^2 u}{\partial t^2} = f(x, y, t, u, v, u_t, u_x, u_y, v_x, v_y), \quad (x, y) \in \Omega, \ t > 0, \tag{8.2}$$

along with the initial conditions (2.1)–(2.2) and the simply supported boundary conditions (3.1)–(3.4) may be replaced by

$$u(x, a, t) = a_{10}(x, t), \quad v(x, a, t) = \nabla^2 u(x, a, t) = a_{30}(x, t), \quad a \le x \le b, \ t > 0, \tag{9.1}$$

$$u(x, b, t) = a_{11}(x, t), \quad v(x, b, t) = \nabla^2 u(x, b, t) = a_{31}(x, t), \quad a \le x \le b, \ t > 0, \tag{9.2}$$

$$u(a, y, t) = b_{10}(y, t), \quad v(a, y, t) = \nabla^2 u(a, y, t) = b_{03}(y, t), \quad a \le y \le b, \ t > 0, \tag{9.3}$$

$$u(b, y, t) = b_{11}(y, t), \quad v(b, y, t) = \nabla^2 u(b, y, t) = b_{13}(y, t), \quad a \le y \le b, \ t > 0. \tag{9.4}$$

Using coupled approach, the fourth-order PDE (1) is expressed as a system of two second-order PDEs (8.1)–(8.2) and the considered boundary conditions (3.1)–(3.4) split into Dirichlet boundary conditions (9.1)–(9.4) for u and its Laplacian v. Hence, no fictitious points for incorporating the boundary conditions are needed.

The exact and the approximate solution values of $u(x, y, t)$ and $v(x, y, t)$ at the grid point (x_l, y_m, t_j) are denoted by $U^j_{l,m}$, $u^j_{l,m}$, $V^j_{l,m}$ and $v^j_{l,m}$, respectively.

At the grid point (x_l, y_m, t_j), for $W = A, B, C, D, U, V$ and f we denote

$$W_{abc} = \Big(\frac{\partial^{a+b+c} W}{\partial x^a \partial y^b \partial t^c}\Big)^j_{l,m}, \quad a, b, c = 0, 1, 2, \dots,$$

$$\alpha^j_{l,m} = \Big(\frac{\partial f}{\partial U}\Big)^j_{l,m}, \quad \beta^j_{l,m} = \Big(\frac{\partial f}{\partial V}\Big)^j_{l,m}, \quad \gamma^j_{l,m} = \Big(\frac{\partial f}{\partial U_x}\Big)^j_{l,m}, \quad \delta^j_{l,m} = \Big(\frac{\partial f}{\partial V_x}\Big)^j_{l,m},$$

$$\zeta^j_{l,m} = \Big(\frac{\partial f}{\partial U_y}\Big)^j_{l,m}, \quad \eta^j_{l,m} = \Big(\frac{\partial f}{\partial V_y}\Big)^j_{l,m}, \quad \xi^j_{l,m} = \Big(\frac{\partial f}{\partial t}\Big)^j_{l,m}.$$

Further, at the grid point (x_l, y_m, t_j), the exact solutions $U^j_{l,m}$ and $V^j_{l,m}$ satisfies

$$U_{200} + U_{020} = V_{000},$$

$$A_{000}(V_{200} + V_{020}) + U_{002} = f(x_l, y_m, t_j, U^j_{l,m}, V^j_{l,m}, U^j_{t_{l,m}}, U^j_{x_{l,m}}, U^j_{y_{l,m}}, V^j_{x_{l,m}}, V^j_{y_{l,m}}) \equiv F^j_{l,m} \quad \text{(say)}.$$

We define the following discrete operators:

$$\delta_x U^j_{l,m} = U^j_{l+\frac12,m} - U^j_{l-\frac12,m}, \quad \mu_x U^j_{l,m} = \frac12(U^j_{l+\frac12,m} + U^j_{l-\frac12,m}),$$

$$\delta_y U^j_{l,m} = U^j_{l,m+\frac12} - U^j_{l,m-\frac12}, \quad \mu_y U^j_{l,m} = \frac12(U^j_{l,m+\frac12} + U^j_{l,m-\frac12}),$$

$$\delta_t U^j_{l,m} = U^{j+\frac12}_{l,m} - U^{j-\frac12}_{l,m}, \quad \mu_t U^j_{l,m} = \frac12(U^{j+\frac12}_{l,m} + U^{j-\frac12}_{l,m}),$$

where δ_x and μ_x are second-order central difference and averaging operators with respect to x-direction, respectively, etc.

Now for $p, q = 0, \pm1$, we define the following approximations to u, u_t, u_{tt}, u_x, u_y, u_{xx} and u_{yy} at grid points (x_{l+p}, y_{m+q}, t_j) as under:

$$\overline{U}^j_{l+p,m+q} = \theta U^{j+1}_{l+p,m+q} + (1 - 2\theta)U^j_{l+p,m+q} + \theta U^{j-1}_{l+p,m+q}, \quad 0 < \theta < 1, \tag{10.1}$$

$$\overline{U}^j_{t_{l+p,m+q}} = (U^{j+1}_{l+p,m+q} - U^{j-1}_{l+p,m+q})\frac{1}{2k}, \tag{10.2}$$

$$\overline{U}^j_{tt_{l+p,m+q}} = (U^{j+1}_{l+p,m+q} - 2U^j_{l+p,m+q} + U^{j-1}_{l+p,m+q})\frac{1}{k^2}, \tag{10.3}$$

$$\overline{U}^j_{x_{l,m+q}} = (\overline{U}^j_{l+1,m+q} - \overline{U}^j_{l-1,m+q})\frac{1}{2h}, \tag{10.4}$$
$$\overline{U}^j_{x_{l\pm1,m+q}} = (\pm3\overline{U}^j_{l\pm1,m+q} \mp 4\overline{U}^j_{l,m+q} \pm \overline{U}^j_{l\mp1,m+q})\frac{1}{2h}, \tag{10.5}$$
$$\overline{U}^j_{y_{l+p,m}} = (\overline{U}^j_{l+p,m+1} - \overline{U}^j_{l+p,m-1})\frac{1}{2h}, \tag{10.6}$$
$$\overline{U}^j_{y_{l+p,m\pm1}} = (\pm3\overline{U}^j_{l+p,m\pm1} \mp 4\overline{U}^j_{l+p,m} \pm \overline{U}^j_{l+p,m\mp1})\frac{1}{2h}, \tag{10.7}$$
$$\overline{U}^j_{xx_{l,m+q}} = (\overline{U}^j_{l+1,m+q} - 2\overline{U}^j_{l,m+q} + \overline{U}^j_{l-1,m+q})\frac{1}{h^2}, \tag{10.8}$$
$$\overline{U}^j_{yy_{l+p,m}} = (\overline{U}^j_{l+p,m+1} - 2\overline{U}^j_{l+p,m} + \overline{U}^j_{l+p,m-1})\frac{1}{h^2}, \tag{10.9}$$

and similar approximations are defined for v, v_x, v_y, v_{xx} and v_{yy} at the grid points (x_{l+p}, y_{m+q}, t_j).

A cental difference scheme based on central difference approximations for (P1) may be written in a coupled form as

$$(\delta_x^2 + \delta_y^2)\overline{U}^j_{l,m} = h^2\overline{V}^j_{l,m} + O(k^2h^2 + h^4), \tag{11.1}$$
$$A_{000}(\delta_x^2 + \delta_y^2)\overline{V}^j_{l,m} + h^2\overline{U}^j_{tt_{l,m}} = h^2\overline{F}^j_{l,m} + O(k^2h^2 + h^4), \quad l, m = 1, \dots, N,\ j = 1, \dots, J, \tag{11.2}$$

where

$$\overline{F}^j_{l,m} = f(x_l, y_m, t_j, \overline{U}^j_{l,m}, \overline{V}^j_{l,m}, \overline{U}^j_{t_{l,m}}, \overline{U}^j_{x_{l,m}}, \overline{V}^j_{x_{l,m}}, \overline{U}^j_{y_{l,m}}, \overline{V}^j_{y_{l,m}}).$$

Note that the difference scheme (11.1)–(11.2) is of $O(k^2 + h^2)$ and requires only five spatial grid points of a single computational stencil at each time level.

Using the approximations (10.1)–(10.9), we define

$$\begin{aligned}\overline{F}^j_{l\pm1,m} &= f(x_{l\pm1}, y_m, t_j, \overline{U}^j_{l\pm1,m}, \overline{V}^j_{l\pm1,m}, \overline{U}^j_{t_{l\pm1,m}}, \overline{U}^j_{x_{l\pm1,m}}, \overline{V}^j_{x_{l\pm1,m}}, \overline{U}^j_{y_{l\pm1,m}}, \overline{V}^j_{y_{l\pm1,m}})\\ &= F^j_{l\pm1,m} + \frac{k^2}{6}T_1 - \frac{h^2}{3}T_2 + \frac{h^2}{6}T_3 + O(\pm h^3 \pm k^2h + k^2h^2),\end{aligned} \tag{12.1}$$
$$\begin{aligned}\overline{F}^j_{l,m\pm1} &= f(x_l, y_{m\pm1}, t_j, \overline{U}^j_{l,m\pm1}, \overline{V}^j_{l,m\pm1}, \overline{U}^j_{t_{l,m\pm1}}, \overline{U}^j_{x_{l,m\pm1}}, \overline{V}^j_{x_{l,m\pm1}}, \overline{U}^j_{y_{l,m\pm1}}, \overline{V}^j_{y_{l,m\pm1}})\\ &= F^j_{l,m\pm1} + \frac{k^2}{6}T_1 - \frac{h^2}{3}T_3 + \frac{h^2}{6}T_2 + O(\pm h^3 \pm k^2h + k^2h^2),\end{aligned} \tag{12.2}$$

where

$$\begin{aligned}T_1 &= 6\theta(U_{002}\alpha^j_{l,m} + V_{002}\beta^j_{l,m} + U_{102}\gamma^j_{l,m} + V_{102}\delta^j_{l,m} + U_{012}\zeta^j_{l,m} + V_{012}\eta^j_{l,m}) + U_{003}\xi^j_{l,m},\\ T_2 &= U_{300}\gamma^j_{l,m} + V_{300}\delta^j_{l,m},\\ T_3 &= U_{030}\zeta^j_{l,m} + V_{030}\eta^j_{l,m}.\end{aligned}$$

In order to achieve high order accuracy of the first order partial derivatives of the solution variables, consider the following linear combinations:

$$\overline{\overline{U}}^j_{x_{l,m}} = \overline{U}^j_{x_{l,m}} + h[a_1(\overline{V}^j_{l+1,m} - \overline{V}^j_{l-1,m}) + a_2(\overline{U}^j_{yy_{l+1,m}} - \overline{U}^j_{yy_{l-1,m}})], \tag{13.1}$$
$$\overline{\overline{U}}^j_{y_{l,m}} = \overline{U}^j_{y_{l,m}} + h[b_1(\overline{V}^j_{l,m+1} - \overline{V}^j_{l,m-1}) + b_2(\overline{U}^j_{xx_{l,m+1}} - \overline{U}^j_{xx_{l,m-1}})], \tag{13.2}$$
$$\begin{aligned}\overline{\overline{V}}^j_{x_{l,m}} &= \overline{V}^j_{x_{l,m}} + h[c_1\{(\overline{F}^j_{l+1,m} - \overline{F}^j_{l-1,m}) - (\overline{U}^j_{tt_{l+1,m}} - \overline{U}^j_{tt_{l-1,m}})\} + c_2(\overline{V}^j_{yy_{l+1,m}} - \overline{V}^j_{yy_{l-1,m}})]\\ &\quad + h^2[c_3\overline{V}^j_{xx_{l,m}} + c_4\overline{V}^j_{yy_{l,m}}],\end{aligned} \tag{13.3}$$
$$\begin{aligned}\overline{\overline{V}}^j_{y_{l,m}} &= \overline{V}^j_{y_{l,m}} + h[d_1\{(\overline{F}^j_{l,m+1} - \overline{F}^j_{l,m-1}) - (\overline{U}^j_{tt_{l,m+1}} - \overline{U}^j_{tt_{l,m-1}})\} + d_2(\overline{V}^j_{xx_{l,m+1}} - \overline{V}^j_{xx_{l,m-1}})]\\ &\quad + h^2[d_3\overline{V}^j_{xx_{l,m}} + d_4\overline{V}^j_{yy_{l,m}}],\end{aligned} \tag{13.4}$$

where a_ks, b_ks, c_ks and d_ks are the parameters to be suitably determined.

Finally, we apply the above additional approximations to the first order derivatives to obtain the functional value at the grid point (x_l, y_m, t_j), defined by the relation

$$\overline{\overline{F}}^j_{l,m} = f(x_l, y_m, t_j, \overline{U}^j_{l,m}, \overline{V}^j_{l,m}, \overline{U}^j_{t_{l,m}}, \overline{\overline{U}}^j_{x_{l,m}}, \overline{\overline{V}}^j_{x_{l,m}}, \overline{\overline{U}}^j_{y_{l,m}}, \overline{\overline{V}}^j_{y_{l,m}}). \tag{14}$$

Then, the required numerical method for (P1) in compact operator form may be written as

$$[6(\delta_x^2 + \delta_y^2) + \delta_x^2\delta_y^2]\overline{U}^j_{l,m} = \frac{h^2}{2}[12 + \delta_x^2 + \delta_y^2]\overline{V}^j_{l,m} + O(k^2h^4 + h^6), \tag{15.1}$$

$$\begin{aligned}
&[L_1(\delta_x^2 + \delta_y^2) + L_2\delta_x^2\delta_y^2]\overline{V}^j_{l,m} + \frac{h^2}{2}[J_1\overline{U}^j_{tt_{l+1,m}} + J_2\overline{U}^j_{tt_{l-1,m}} + J_3\overline{U}^j_{tt_{l,m+1}} + J_4\overline{U}^j_{tt_{l,m-1}} + 8\overline{U}^j_{tt_{l,m}}]\\
&\quad = \frac{h^2}{2}[J_1\overline{F}^j_{l+1,m} + J_2\overline{F}^j_{l-1,m} + J_3\overline{F}^j_{l,m+1} + J_4\overline{F}^j_{l,m-1} + 8\overline{\overline{F}}^j_{l,m}] + \overline{T}^j_{l,m}, \quad l, m = 1, \dots, N,\ j = 1, \dots, J,
\end{aligned} \tag{15.2}$$

where the local truncation error $\overline{T}^j_{l,m} = O(k^2h^2 + h^6)$, implying that the accuracy of the method is $O(k^2 + h^4)$ and

$$L_1 = 6A_{000} + \frac{h^2}{2}\Big(A_{200} + A_{020} - 2\frac{A_{100}}{A_{000}}A_{100} - 2\frac{A_{010}}{A_{000}}A_{010}\Big), \quad L_2 = A_{000},$$

$$J_1 = \Big(1 - \frac{hA_{100}}{A_{000}}\Big), \quad J_2 = \Big(1 + \frac{hA_{100}}{A_{000}}\Big), \quad J_3 = \Big(1 - \frac{hA_{010}}{A_{000}}\Big), \quad J_4 = \Big(1 + \frac{hA_{010}}{A_{000}}\Big).$$

When $k \propto h^2$, that is, for a fixed value of mesh parameter λ, method (15.1)–(15.2) is fourth-order accurate in spatial dimension.

Using Taylor series expansion and differentiating equations (8.1)–(8.2) twice with respect to x and y, we get

$$[6\delta_x^2 + 6\delta_y^2 + \delta_x^2\delta_y^2]U^j_{l,m} = \frac{h^2}{2}[12 + \delta_x^2 + \delta_y^2]V^j_{l,m} + O(h^6), \tag{16.1}$$

$$\begin{aligned}
&[L_1(\delta_x^2 + \delta_y^2) + L_2\delta_x^2\delta_y^2]V^j_{l,m} + \frac{h^2}{2}[J_1U^j_{tt_{l+1,m}} + J_2U^j_{tt_{l-1,m}} + J_3U^j_{tt_{l,m+1}} + J_4U^j_{tt_{l,m-1}} + 8U^j_{tt_{l,m}}]\\
&\quad = \frac{h^2}{2}[J_1F^j_{l+1,m} + J_2F^j_{l-1,m} + J_3F^j_{l,m+1} + J_4F^j_{l,m-1} + 8F^j_{l,m}] + O(h^6).
\end{aligned} \tag{16.2}$$

Again invoking series expansion, we have

$$\begin{aligned}
&[L_1(\delta_x^2 + \delta_y^2) + L_2\delta_x^2\delta_y^2]\overline{V}^j_{l,m} + \frac{h^2}{2}[J_1\overline{U}^j_{tt_{l+1,m}} + J_2\overline{U}^j_{tt_{l-1,m}} + J_3\overline{U}^j_{tt_{l,m+1}} + J_4\overline{U}^j_{tt_{l,m-1}} + 8\overline{U}^j_{tt_{l,m}}]\\
&\quad = [L_1(\delta_x^2 + \delta_y^2) + L_2\delta_x^2\delta_y^2]V^j_{l,m} + \frac{h^2}{2}[J_1U^j_{tt_{l+1,m}} + J_2U^j_{tt_{l-1,m}} + J_3U^j_{tt_{l,m+1}} + J_4U^j_{tt_{l,m-1}} + 8U^j_{tt_{l,m}}]\\
&\qquad + 6\theta k^2h^2(V_{202} + V_{022}) + \frac{k^2h^2}{2}U_{004} + O(k^2h^4 + k^4).
\end{aligned} \tag{17}$$

By using the approximations (10.1)–(10.9) and (12.1)–(12.2), from (13.1)–(13.4), we get

$$\overline{\overline{U}}^j_{x_{l,m}} = U^j_{x_{l,m}} + \theta k^2U_{102} + \frac{h^2}{6}T_4 + O(k^2h^2 + h^4), \tag{18.1}$$

$$\overline{\overline{U}}^j_{y_{l,m}} = U^j_{y_{l,m}} + \theta k^2U_{012} + \frac{h^2}{6}T_5 + O(k^2h^2 + h^4), \tag{18.2}$$

$$\overline{\overline{V}}^j_{x_{l,m}} = V^j_{x_{l,m}} + \theta k^2V_{102} + \frac{h^2}{6}T_6 + O(k^2h^2 + h^4), \tag{18.3}$$

$$\overline{\overline{V}}^j_{y_{l,m}} = V^j_{y_{l,m}} + \theta k^2V_{012} + \frac{h^2}{6}T_7 + O(k^2h^2 + h^4), \tag{18.4}$$

where

$$\begin{aligned}
T_4 &= (1 + 12a_1)U_{300} + 12(a_1 + a_2)U_{120},\\
T_5 &= (1 + 12b_1)U_{030} + 12(b_1 + b_2)U_{210},\\
T_6 &= (1 + 12c_1A_{000})V_{300} + 12(c_1A_{000} + c_2)V_{120} + (12c_1A_{100} + 6c_3)V_{200} + (12c_1A_{100} + 6c_4)V_{020},\\
T_7 &= (1 + 12d_1A_{000})V_{030} + 12(d_1A_{000} + d_2)V_{210} + (12d_1A_{010} + 6d_3)V_{200} + (12d_1A_{010} + 6d_4)V_{020}.
\end{aligned}$$

By the help of approximations (18.1)–(18.4), from (14) we obtain

$$\overline{\overline{F}}{}^j_{l,m} = F^j_{l,m} + \frac{k^2}{6}T_1 + \frac{h^2}{6}T_8 + O(k^2h^2 + h^4), \tag{19}$$

where

$$T_8 = T_4\gamma^j_{l,m} + T_5\zeta^j_{l,m} + T_6\delta^j_{l,m} + T_7\eta^j_{l,m}.$$

Now using the approximations (12.1)–(12.2), (19) on the right-hand side, (17) on the left-hand side of (15.2) and by the help of relation (16.2), we obtain the local truncation error $\overline{T}^j_{l,m}$ associated with (15.2) as:

$$\overline{T}^j_{l,m} = \frac{h^4}{6}(4T_8 - T_2 - T_3) + k^2h^2\Big(T_1 - 6\theta A_{000}(V_{202} + V_{022}) - \frac{1}{2}U_{004}\Big) + O(k^2h^4 + k^4h^2 + h^6). \tag{20}$$

In order to obtain the difference method (15.2) of $O(k^2 + h^4)$, the coefficient of h^4 in (20) must be zero, which gives the values of parameters

$$a_1 = b_1 = -\frac{1}{16}, \quad a_2 = b_2 = \frac{1}{16}, \qquad c_1 = d_1 = -\frac{1}{16A_{000}},$$
$$c_2 = d_2 = \frac{1}{16}, \quad c_3 = c_4 = \frac{A_{100}}{8A_{000}}, \quad d_3 = d_4 = \frac{A_{010}}{8A_{000}}.$$

Thus, for the above set of parameters, $\overline{T}^j_{l,m} = O(k^2h^2 + h^6)$ and hence the difference scheme (15.1)–(15.2) is of $O(k^2 + h^4)$, for arbitrary θ.

Now we consider the two-space-dimensional quasi-linear fourth-order parabolic PDE (1), i.e. when $A = A(x, y, t, u, \nabla^2 u)$. When the coefficient A is a function of not only the independent variables x, y and t but also of the dependent variables u and v, the difference scheme (15.2) needs to be modified. For this reason, we use the following central differences:

$$A^j_{x_{l,m}} = (\overline{A}^j_{l+1,m} - \overline{A}^j_{l-1,m})/(2h), \tag{21.1}$$
$$A^j_{y_{l,m}} = (\overline{A}^j_{l,m+1} - \overline{A}^j_{l,m-1})/(2h), \tag{21.2}$$
$$A^j_{xx_{l,m}} = (\overline{A}^j_{l+1,m} - 2\overline{A}^j_{l,m} + \overline{A}^j_{l-1,m})/h^2, \tag{21.3}$$
$$A^j_{yy_{l,m}} = (\overline{A}^j_{l,m+1} - 2\overline{A}^j_{l,m} + \overline{A}^j_{l,m-1})/h^2. \tag{21.4}$$

where

$$\overline{A}^j_{l,m} = A(x_l, y_m, t_j, \overline{U}^j_{l,m}, \overline{V}^j_{l,m}),$$
$$\overline{A}^j_{l\pm1,m} = A(x_{l\pm1}, y_m, t_j, \overline{U}^j_{l\pm1,m}, \overline{V}^j_{l\pm1,m}),$$
$$\overline{A}^j_{l,m\pm1} = A(x_l, y_{m\pm1}, t_j, \overline{U}^j_{l,m\pm1}, \overline{V}^j_{l,m\pm1}).$$

Using the central difference approximations (21.1)–(21.4) in the difference scheme (15.1)–(15.2), the required numerical method of $O(k^2 + h^4)$ for the two-dimensional quasi-linear PDE (1) is attained.

The three-level implicit difference scheme proposed here in a coupled form is based on only (9 + 9 + 9) grid points and no fictitious points for handling the boundary conditions are used. Incorporating the initial and boundary conditions given by (2.1)–(2.2) and (3.1)–(3.4), method (15.1)–(15.2) results in a block tri-diagonal matrix system. If the differential equation (1) is linear, we can solve the linear system using the Gauss–Seidel method; in the nonlinear case, we can use Newton's nonlinear block iteration procedure (see Kelly [15]).

3 Two-Level Compact Operator Method for (P2)

In this section, we present a finite difference approximation for (P2) which retains the compact stencil of nine points without any special treatment at the boundary.

Firstly, we consider $B = B(x, y, t)$. Let us decompose problem (P2) as an elliptic-parabolic pair to obtain a system of two second-order PDEs in the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = v, \qquad (x, t) \in \Omega, \tag{22.1}$$

$$B(x, y, t)\Big(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\Big) + \frac{\partial u}{\partial t} = g(x, y, t, u, v, u_x, u_y, v_x, v_y), \quad (x, t) \in \Omega, \ t > 0, \tag{22.2}$$

subject to the initial condition (2.1) and boundary conditions (3.1)–(3.4).

Note that using successive derivative values along the boundary, u and v are known on all sides of the solution region Ω.

To determine the numerical solution of the proposed initial boundary value problem (P2), we discretize the solution domain using the mesh defined in Section 2 and at the grid point (x_l, y_m, t_j), we let

$$\rho^j_{l,m} = \Big(\frac{\partial g}{\partial U}\Big)^j_{l,m}, \quad \sigma^j_{l,m} = \Big(\frac{\partial g}{\partial V}\Big)^j_{l,m}, \quad \phi^j_{l,m} = \Big(\frac{\partial g}{\partial U_x}\Big)^j_{l,m}, \quad \tau^j_{l,m} = \Big(\frac{\partial g}{\partial V_x}\Big)^j_{l,m},$$
$$\chi^j_{l,m} = \Big(\frac{\partial g}{\partial U_y}\Big)^j_{l,m}, \quad \omega^j_{l,m} = \Big(\frac{\partial g}{\partial V_y}\Big)^j_{l,m}, \quad \varepsilon^j_{l,m} = \Big(\frac{\partial g}{\partial t}\Big)^j_{l,m}.$$

Further, at the grid point (x_l, y_m, t_j), the differential equations (22.1)–(22.2) can be written as

$$U_{200} + U_{020} = V_{000},$$
$$A_{000}(V_{200} + V_{020}) + U_{001} = g(x_l, y_m, t_j, U^j_{l,m}, V^j_{l,m}, U^j_{x_{l,m}}, U^j_{y_{l,m}}, V^j_{x_{l,m}}, V^j_{y_{l,m}}) \equiv G^j_{l,m} \quad \text{(say)}.$$

Let

$$\hat{t}_j = t_j + \frac{k}{2}.$$

In order to derive the high accuracy method, we need the following approximations: For $r, s = 0, \pm 1$ and for $S = U, V$, we denote

$$\hat{S}^j_{l+r,m+s} = (S^{j+1}_{l+r,m+s} + S^j_{l+r,m+s})\frac{1}{2}, \tag{23.1}$$

$$\hat{S}^j_{t_{l+r,m+s}} = (S^{j+1}_{l+r,m+s} - S^j_{l+r,m+s})\frac{1}{k}, \tag{23.2}$$

$$\hat{S}^j_{x_{l,m+s}} = (\hat{S}^j_{l+1,m+s} - \hat{S}^j_{l-1,m+s})\frac{1}{2h}, \tag{23.3}$$

$$\hat{S}^j_{x_{l\pm1,m+s}} = (\pm 3\hat{S}^j_{l\pm1,m+s} \mp 4\hat{S}^j_{l,m+s} \pm \hat{S}^j_{l\mp1,m+s})\frac{1}{2h}, \tag{23.4}$$

$$\hat{S}^j_{y_{l+r,m}} = (\hat{S}^j_{l+r,m+1} - \hat{S}^j_{l+r,m-1})\frac{1}{2h}, \tag{23.5}$$

$$\hat{S}^j_{y_{l+r,m\pm1}} = (\pm 3\hat{S}^j_{l+r,m\pm1} \mp 4\hat{S}^j_{l+r,m} \pm \hat{S}^j_{l+r,m\mp1})\frac{1}{2h}, \tag{23.6}$$

$$\hat{S}^j_{xx_{l,m+s}} = (\hat{S}^j_{l+1,m+s} - 2\hat{S}^j_{l,m+s} + \hat{S}^j_{l-1,m+s})\frac{1}{h^2}, \tag{23.7}$$

$$\hat{S}^j_{yy_{l+r,m}} = (\hat{S}^j_{l+r,m+1} - 2\hat{S}^j_{l+r,m} + \hat{S}^j_{l+r,m-1})\frac{1}{h^2}, \tag{23.8}$$

and $\hat{B}^j_{l,m} = B(x_l, y_m, \hat{t}_j)$, $\hat{B}^j_{x_{l,m}} = B_x(x_l, y_m, \hat{t}_j)$, etc. The five-point central difference approximations for (P2) can be written in operator form as follows:

$$(\delta_x^2 + \delta_y^2)\mu_t U^{j+\frac{1}{2}}_{l,m} = h^2 \mu_t V^{j+\frac{1}{2}}_{l,m} + O(k^2h^2 + h^4), \tag{24.1}$$

$$\hat{B}^j_{l,m}(\delta_x^2 + \delta_y^2)\mu_t V^{j+\frac{1}{2}}_{l,m} + \frac{h^2}{k}\delta_t U^{j+\frac{1}{2}}_{l,m} = h^2 \hat{G}^j_{l,m} + O(k^2h^2 + h^4), \quad l, m = 1, \dots, N, \ j = 0, \dots, J, \tag{24.2}$$

where

$$\hat{G}^j_{l,m} = g(x_l, y_m, \hat{t}_j, \hat{U}^j_{l,m}, \hat{V}^j_{l,m}, \hat{U}^j_{x_{l,m}}, \hat{V}^j_{x_{l,m}}, \hat{U}^j_{y_{l,m}}, \hat{V}^j_{y_{l,m}}).$$

By the using the approximations (23.1)–(23.8), we obtain

$$\begin{aligned}\widehat{G}^j_{l\pm1,m} &= g(x_{l\pm1}, y_m, \widehat{t}_j, \widehat{U}^j_{l\pm1,m}, \widehat{V}^j_{l\pm1,m}, \widehat{U}^j_{x_{l\pm1,m}}, \widehat{V}^j_{x_{l\pm1,m}}, \widehat{U}^j_{y_{l\pm1,m}}, \widehat{V}^j_{y_{l\pm1,m}})\\ &= G^j_{l\pm1,m} + \frac{k}{2}S_1 - \frac{h^2}{3}S_2 + \frac{h^2}{6}S_3 + O(\pm hk \pm h^3 + k^2 + kh^2 + h^4), \end{aligned} \tag{25.1}$$

$$\begin{aligned}\widehat{G}^j_{l,m\pm1} &= g(x_l, y_{m\pm1}, \widehat{t}_j, \widehat{U}^j_{l,m\pm1}, \widehat{V}^j_{l,m\pm1}, \widehat{U}^j_{x_{l,m\pm1}}, \widehat{V}^j_{x_{l,m\pm1}}, \widehat{U}^j_{y_{l,m\pm1}}, \widehat{V}^j_{y_{l,m\pm1}})\\ &= G^j_{l,m\pm1} + \frac{k}{2}S_1 - \frac{h^2}{3}S_3 + \frac{h^2}{6}S_2 + O(\pm hk \pm h^3 + k^2 + kh^2 + h^4), \end{aligned} \tag{25.2}$$

where

$$\begin{aligned} S_1 &= U_{001}\rho^j_{l,m} + V_{001}\sigma^j_{l,m} + U_{101}\phi^j_{l,m} + V_{101}\tau^j_{l,m} + U_{011}\chi^j_{l,m} + V_{011}\omega^j_{l,m} + \varepsilon^j_{l,m},\\ S_2 &= U_{300}\phi^j_{l,m} + V_{300}\tau^j_{l,m},\\ S_3 &= U_{030}\chi^j_{l,m} + V_{030}\omega^j_{l,m}. \end{aligned}$$

Next we consider the following approximations:

$$\widehat{\widehat{U}}^j_{x_{l,m}} = \widehat{U}^j_{x_{l,m}} + h[p_1(\widehat{V}^j_{l+1,m} - \widehat{V}^j_{l-1,m}) + p_2(\widehat{U}^j_{yy_{l+1,m}} - \widehat{U}^j_{yy_{l-1,m}})], \tag{26.1}$$

$$\widehat{\widehat{U}}^j_{y_{l,m}} = \widehat{U}^j_{y_{l,m}} + h[q_1(\widehat{V}^j_{l,m+1} - \widehat{V}^j_{l,m-1}) + q_2(\widehat{U}^j_{xx_{l,m+1}} - \widehat{U}^j_{xx_{l,m-1}})], \tag{26.2}$$

$$\begin{aligned}\widehat{\widehat{V}}^j_{x_{l,m}} &= \widehat{V}^j_{x_{l,m}} + h[r_1\{(\widehat{G}^j_{l+1,m} - \widehat{G}^j_{l-1,m}) - (\widehat{U}^j_{t_{l+1,m}} - \widehat{U}^j_{t_{l-1,m}})\} + r_2(\widehat{V}^j_{yy_{l+1,m}} - \widehat{V}^j_{yy_{l-1,m}})]\\ &\quad + h^2[r_3\widehat{V}^j_{xx_{l,m}} + r_4\widehat{V}^j_{yy_{l,m}}], \end{aligned} \tag{26.3}$$

$$\begin{aligned}\widehat{\widehat{V}}^j_{y_{l,m}} &= \widehat{V}^j_{y_{l,m}} + h[s_1\{(\widehat{G}^j_{l,m+1} - \widehat{G}^j_{l,m-1}) - (\widehat{U}^j_{t_{l,m+1}} - \widehat{U}^j_{t_{l,m-1}})\} + s_2(\widehat{V}^j_{xx_{l,m+1}} - \widehat{V}^j_{xx_{l,m-1}})]\\ &\quad + h^2[s_3\widehat{V}^j_{xx_{l,m}} + s_4\widehat{V}^j_{yy_{l,m}}], \end{aligned} \tag{26.4}$$

where p_i, q_i, $i = 1, 2$ and r_j, s_j, $j = 1, \ldots, 4$ are free parameters to be determined. With the help of the approximations (23.1)–(23.8) and (25.1)–(25.2), from (26.1)–(26.4), we get

$$\widehat{\widehat{U}}^j_{x_{l,m}} = U^j_{x_{l,m}} + \frac{k}{2}U_{101} + \frac{h^2}{6}S_4 + O(k^2 + kh^2 + h^4), \tag{27.1}$$

$$\widehat{\widehat{U}}^j_{y_{l,m}} = U^j_{y_{l,m}} + \frac{k}{2}U_{011} + \frac{h^2}{6}S_5 + O(k^2 + kh^2 + h^4), \tag{27.2}$$

$$\widehat{\widehat{V}}^j_{x_{l,m}} = V^j_{x_{l,m}} + \frac{k}{2}V_{101} + \frac{h^2}{6}S_6 + O(k^2 + kh^2 + h^4), \tag{27.3}$$

$$\widehat{\widehat{V}}^j_{y_{l,m}} = V^j_{y_{l,m}} + \frac{k}{2}V_{011} + \frac{h^2}{6}S_7 + O(k^2 + kh^2 + h^4), \tag{27.4}$$

where

$$\begin{aligned} S_4 &= (1 + 12p_1)U_{300} + 12(p_1 + p_2)U_{120},\\ S_5 &= (1 + 12q_1)U_{030} + 12(q_1 + q_2)U_{210},\\ S_6 &= (1 + 12r_1\widehat{B}^j_{l,m})V_{300} + 12(r_1\widehat{B}^j_{l,m} + r_2)V_{120} + (12r_1\widehat{B}^j_{x_{l,m}} + 6r_3)V_{200} + (12r_1\widehat{B}^j_{x_{l,m}} + 6r_4)V_{020},\\ S_7 &= (1 + 12s_1\widehat{B}^j_{l,m})V_{030} + 12(s_1\widehat{B}^j_{l,m} + d_2)V_{210} + (12s_1\widehat{B}^j_{x_{l,m}} + 6s_3)V_{200} + (12s_1\widehat{B}^j_{y_{l,m}} + 6s_4)V_{020}. \end{aligned}$$

Finally, we define

$$\widehat{\widehat{G}}^j_{l,m} = g(x_l, y_m, \widehat{t}_j, \widehat{U}^j_{l,m}, \widehat{V}^j_{l,m}, \widehat{\widehat{U}}^j_{x_{l,m}}, \widehat{\widehat{V}}^j_{x_{l,m}}, \widehat{\widehat{U}}^j_{y_{l,m}}, \widehat{\widehat{V}}^j_{y_{l,m}}).$$

By the help of the approximations (27.1)–(27.4), we get

$$\widehat{\widehat{G}}^j_{l,m} = G^j_{l,m} + \frac{k}{2}S_1 + \frac{h^2}{6}S_8 + O(k^2 + kh^2 + h^4), \tag{28}$$

where

$$S_8 = S_4\phi^j_{l,m} + S_5\chi^j_{l,m} + S_6\tau^j_{l,m} + S_7\omega^j_{l,m}.$$

Then at each grid point (x_l, y_m, t_j), $l, m = 1, \dots, N$, $j = 0, \dots, J$, we may write the compact operator method of $O(k^2 + kh^2 + h^4)$ for the solution of differential equations (22.1)–(22.2) as follows:

$$[6(\delta_x^2 + \delta_y^2) + \delta_x^2\delta_y^2]\mu_t U^{j+\frac{1}{2}}_{l,m} = \frac{h^2}{2}[12 + \delta_x^2 + \delta_y^2]\mu_t V^{j+\frac{1}{2}}_{l,m} + O(k^2h^2 + kh^4 + h^6), \tag{29.1}$$

$$[I_1(\delta_x^2 + \delta_y^2) + I_2\delta_x^2\delta_y^2]\mu_t V^{j+\frac{1}{2}}_{l,m} + \frac{h^2}{2k}\delta_t[R_1 U^{j+\frac{1}{2}}_{l+1,m} + R_2 U^{j+\frac{1}{2}}_{l-1,m} + R_3 U^{j+\frac{1}{2}}_{l,m+1} + R_4 U^{j+\frac{1}{2}}_{l,m-1} + 8U^{j+\frac{1}{2}}_{l,m}]$$
$$= \frac{h^2}{2}[R_1\widehat{F}^j_{l+1,m} + R_2\widehat{F}^j_{l-1,m} + R_3\widehat{F}^j_{l,m+1} + R_4\widehat{F}^j_{l,m-1} + 8\widehat{\widehat{F}}^j_{l,m}] + \widehat{T}^j_{l,m}, \tag{29.2}$$

where $\widehat{T}^j_{l,m} = O(k^2h^2 + kh^4 + h^6)$ and

$$I_1 = 6\widehat{B}^j_{l,m} + \frac{h^2}{2}\left(\widehat{B}^j_{xx_{l,m}} + \widehat{B}^j_{yy_{l,m}} - 2\frac{\widehat{B}^j_{x_{l,m}}}{\widehat{B}^j_{l,m}}\widehat{B}^j_{x_{l,m}} - 2\frac{\widehat{B}^j_{y_{l,m}}}{\widehat{B}^j_{l,m}}\widehat{B}^j_{y_{l,m}}\right), \quad I_2 = \widehat{B}^j_{l,m},$$

$$R_1 = \left(1 - \frac{h\widehat{B}^j_{x_{l,m}}}{\widehat{B}^j_{l,m}}\right), \quad R_2 = \left(1 + \frac{h\widehat{B}^j_{x_{l,m}}}{\widehat{B}^j_{l,m}}\right), \quad R_3 = \left(1 - \frac{h\widehat{B}^j_{y_{l,m}}}{\widehat{B}^j_{l,m}}\right), \quad R_4 = \left(1 + \frac{h\widehat{B}^j_{y_{l,m}}}{\widehat{B}^j_{l,m}}\right).$$

Now using the approximations (25.1)–(25.2) and (28), the local truncation error $\widehat{T}^j_{l,m}$ associated with (29.2) is obtained as

$$\widehat{T}^j_{l,m} = \frac{h^4}{6}(4S_8 - S_2 - S_3) + O(k^2h^2 + kh^4 + h^6). \tag{30}$$

The proposed method (29.2) to be of $O(k^2 + kh^2 + h^4)$, the coefficient of h^4 in (30) must be zero. This implies $4S_8 - S_2 - S_3 = 0$. Thus we obtain the values of parameters

$$p_1 = q_1 = -\frac{1}{16}, \quad p_2 = q_2 = \frac{1}{16}, \qquad r_1 = s_1 = -\frac{1}{16\widehat{B}^j_{l,m}},$$

$$r_2 = s_2 = \frac{1}{16}, \qquad r_3 = r_4 = \frac{\widehat{B}^j_{x_{l,m}}}{8\widehat{B}^j_{l,m}}, \quad s_3 = s_4 = \frac{\widehat{B}^j_{y_{l,m}}}{8\widehat{B}^j_{l,m}}.$$

Thus we have derived the difference method (29.1)–(29.2) which is of $O(k^2 + kh^2 + h^4)$ for the differential equations (22.1)–(22.2).

For the quasi-linear problem (P2), i.e. when the coefficient $B = B(x, y, t, u, v)$, we need to modify our proposed difference methods (29.1)–(29.2). In this case, we make use of the following approximations in (29.1)–(29.2):

$$\widehat{B}^j_{x_{l,m}} = (\widehat{B}^j_{l+1,m} - \widehat{B}^j_{l-1,m})\frac{1}{2h},$$
$$\widehat{B}^j_{y_{l,m}} = (\widehat{B}^j_{l,m+1} - \widehat{B}^j_{l,m-1})\frac{1}{2h},$$
$$\widehat{B}^j_{xx_{l,m}} = (\widehat{B}^j_{l+1,m} - 2\widehat{B}^j_{l,m} + \widehat{B}^j_{l-1,m})\frac{1}{h^2},$$
$$\widehat{B}^j_{yy_{l,m}} = (\widehat{B}^j_{l,m+1} - 2\widehat{B}^j_{l,m} + \widehat{B}^j_{l,m-1})\frac{1}{h^2}.$$

where

$$\widehat{B}^j_{l,m} = B(x_l, y_m, \widehat{t}_j, \widehat{U}^j_{l,m}, \widehat{V}^j_{l,m}),$$
$$\widehat{B}^j_{l\pm1,m} = B(x_{l\pm1}, y_m, \widehat{t}_j, \widehat{U}^j_{l\pm1,m}, \widehat{V}^j_{l\pm1,m}),$$
$$\widehat{B}^j_{l,m\pm1} = B(x_l, y_{m\pm1}, \widehat{t}_j, \widehat{U}^j_{l,m\pm1}, \widehat{V}^j_{l,m\pm1}).$$

Substituting the above central difference approximations in (29.1)–(29.2), we obtain the required difference scheme of accuracy $O(k^2 + kh^2 + h^4)$ for the solution of the quasi-linear problem (P2). The matrices represented by the formulas (29.1)–(29.2) and (24.1)–(24.2) are block tri-diagonal for which we have used block iterative methods.

4 High Order Difference Schemes for Singular Equations

We consider fourth-order linear PDEs with variable coefficients of types (P1) and (P2) having the form

$$\nabla^4 u + \frac{\partial^2 u}{\partial t^2} + C(x,y)(u_{xxx} + u_{xyy}) = f(x,y,t), \quad 0 < x, y < 1, \ t > 0, \tag{31}$$

and

$$\nabla^4 u + \frac{\partial u}{\partial t} + D(x,y)u_x = g(x,y,t), \quad 0 < x, y < 1, \ t > 0, \tag{32}$$

respectively. The proposed methods (15.1)–(15.2) and (29.1)–(29.2) when applied to the fourth-order equations (31) and (32) results in the following difference schemes:

$$[6(\delta_x^2 + \delta_y^2) + \delta_x^2\delta_y^2]\bar{u}_{l,m}^j = \frac{h^2}{2}[12 + \delta_x^2 + \delta_y^2]\bar{v}_{l,m}^j, \tag{33.1}$$

$$\begin{aligned}[6(\delta_x^2 + \delta_y^2) + \delta_x^2\delta_y^2]\bar{v}_{l,m}^j &+ \frac{h^2}{2}[12 + \delta_x^2 + \delta_y^2]\bar{u}_{tt_{l,m}}^j \\ = \frac{h^2}{2}\big[&\bar{f}_{l+1,m}^j - C_{l+1,m}\bar{v}_{x_{l+1,m}}^j + \bar{f}_{l-1,m}^j - C_{l-1,m}\bar{v}_{x_{l-1,m}}^j + \bar{f}_{l,m+1}^j - C_{l,m+1}\bar{v}_{x_{l,m+1}}^j \\ &+ \bar{f}_{l,m-1}^j - C_{l,m-1}\bar{v}_{x_{l,m-1}}^j + 8(\bar{f}_{l,m}^j - C_{l,m}\bar{\bar{v}}_{x_{l,m}}^j)\big], \quad l,m = 1,\dots,N, \ j = 1,\dots,J,\end{aligned} \tag{33.2}$$

and

$$[6(\delta_x^2 + \delta_y^2) + \delta_x^2\delta_y^2]\mu_t u_{l,m}^{j+\frac{1}{2}} = \frac{h^2}{2}[12 + \delta_x^2 + \delta_y^2]\mu_t v_{l,m}^{j+\frac{1}{2}}, \tag{34.1}$$

$$\begin{aligned}[6(\delta_x^2 + \delta_y^2) + \delta_x^2\delta_y^2]\mu_t v_{l,m}^{j+\frac{1}{2}} &+ \frac{h^2}{2k}[12 + \delta_x^2 + \delta_y^2]\delta_t u_{l,m}^{j+\frac{1}{2}} \\ = \frac{h^2}{2}\big[&\hat{g}_{l+1,m}^j - D_{l+1,m}\hat{u}_{x_{l+1,m}}^j + \hat{g}_{l-1,m}^j - D_{l-1,m}\hat{u}_{x_{l-1,m}}^j + \hat{g}_{l,m+1}^j - D_{l,m+1}\hat{u}_{x_{l,m+1}}^j \\ &+ \hat{g}_{l,m-1}^j - D_{l,m-1}\hat{u}_{x_{l,m-1}}^j + 8(\hat{g}_{l,m}^j - D_{l,m}\hat{\hat{u}}_{x_{l,m}}^j)\big], \quad l,m = 1,\dots,N, \ j = 0,\dots,J,\end{aligned} \tag{34.2}$$

respectively, where for $p, q = 0, \pm 1$ and $R = C, D$ we have $R_{l+p,m} = R(x_{l+p}, y_m)$, $R_{l,m+q} = R(x_l, y_{m+q})$ and $\bar{f}_{l+p,m}^j = f(x_{l+p}, y_m, t_j)$, $\hat{g}_{l+p,m}^j = g(x_{l+p}, y_m, \hat{t}_j)$, etc.

Note that the local truncation errors of the difference schemes (33.1)–(33.2) and (34.1)–(34.2) are of $O(k^2h^2 + h^6)$ and $O(k^2h^2 + kh^4 + h^6)$, respectively. However, the schemes fail to determine the approximate solution if the variable coefficients $C(x,y)$ and $D(x,y)$ contain singular terms. For instance, if $C(x,y) = 1/x$, then $C_{l-1,m} = 1/x_{l-1}$, which cannot be computed for $l = 1$ at the singular point $x_0 = 0$. Likewise, the other coefficient $D(x,y)$ cannot be evaluated at singular points $x_0 = 0$ and $y_0 = 0$, if it has singular terms. We overcome this shortcoming by using the following approximations:

$$C_{l\pm1,m} = C_{000} \pm hC_{100} + \frac{h^2}{2}C_{200} \pm O(h^3), \tag{35.1}$$

$$C_{l,m\pm1} = C_{000} \pm hC_{010} + \frac{h^2}{2}C_{020} \pm O(h^3), \tag{35.2}$$

$$D_{l\pm1,m} = D_{000} \pm hD_{100} + \frac{h^2}{2}D_{200} \pm O(h^3), \tag{35.3}$$

$$D_{l,m\pm1} = D_{000} \pm hD_{010} + \frac{h^2}{2}D_{020} \pm O(h^3), \tag{35.4}$$

$$\bar{f}_{l\pm1,m}^j = f_{000} \pm hf_{100} + \frac{h^2}{2}f_{200} \pm O(h^3), \tag{35.5}$$

$$\bar{f}_{l,m\pm1}^j = f_{000} \pm hf_{010} + \frac{h^2}{2}f_{020} \pm O(h^3), \tag{35.6}$$

$$\hat{g}_{l\pm1,m}^j = \hat{g}_{l,m}^j \pm h\hat{g}_{x_{l,m}}^j + \frac{h^2}{2}\hat{g}_{xx_{l,m}}^j \pm O(h^3), \tag{35.7}$$

$$\hat{g}_{l,m\pm1}^j = \hat{g}_{l,m}^j \pm h\hat{g}_{y_{l,m}}^j + \frac{h^2}{2}\hat{g}_{yy_{l,m}}^j \pm O(h^3). \tag{35.8}$$

Using the approximations (35.1)–(35.8) in the difference schemes (33.1)–(33.2) and (34.1)–(34.2) and neglecting higher order terms, we obtain the following modified difference approximations for the singular equations (31) and (32) in operator form:

$$[6(\delta_x^2+\delta_y^2)+\delta_x^2\delta_y^2]\overline{u}_{l,m}^{j}=\frac{h^2}{2}[12+\delta_x^2+\delta_y^2]\overline{v}_{l,m}^{j}, \tag{36.1}$$

$$\begin{aligned}&[6(\delta_x^2+\delta_y^2)+\delta_x^2\delta_y^2]\overline{v}_{l,m}^{j}+\frac{h^2}{2}[12+\delta_x^2+\delta_y^2+\frac{hC_{000}}{2}(2\mu_x\delta_x)]\overline{u}_{tt_{l,m}}^{j}\\&=\frac{h}{4}[(-12C_{000}-h^2(C_{200}+C_{020}+C_{000}C_{100}))(2\mu_x\delta_x)]\overline{v}_{l,m}^{j}\\&\quad-\frac{h^2}{2}[(2C_{000}+C_{000}C_{000})\delta_x^2]\overline{v}_{l,m}^{j}-\frac{h}{4}C_{000}(\delta_y^2 2\mu_x\delta_x)\overline{v}_{l,m}^{j}-\frac{h^2}{4}C_{010}(4\mu_x\delta_x\mu_y\delta_y)\overline{v}_{l,m}^{j}\\&\quad+\frac{h^2}{2}[12f_{000}+h^2(f_{200}+f_{020}+C_{000}f_{100})],\quad l,m=1,\dots,N,\ j=1,\dots,J,\end{aligned} \tag{36.2}$$

and

$$[6(\delta_x^2+\delta_y^2)+\delta_x^2\delta_y^2]\mu_t u_{l,m}^{j+\frac12}=\frac{h^2}{2}[12+\delta_x^2+\delta_y^2]\mu_t v_{l,m}^{j+\frac12}, \tag{37.1}$$

$$\begin{aligned}&[6(\delta_x^2+\delta_y^2)+\delta_x^2\delta_y^2]\mu_t v_{l,m}^{j+\frac12}+\frac{h^2}{2k}[12+\delta_x^2+\delta_y^2]\delta_t u_{l,m}^{j+\frac12}\\&=\frac{h}{4}D_{000}(2\mu_x\delta_x)\mu_t u_{l,m}^{j+\frac12}+\frac{h}{4}[(-12D_{000}-h^2(D_{200}+D_{020}))(2\mu_x\delta_x)]\mu_t u_{l,m}^{j+\frac12}\\&\quad-h^2D_{100}\delta_x^2\,\mu_t u_{l,m}^{j+\frac12}-\frac{h}{4}D_{000}(\delta_y^2 2\mu_x\delta_x)\mu_t u_{l,m}^{j+\frac12}-\frac{h^2}{4}D_{010}(4\mu_x\delta_x\mu_y\delta_y)\mu_t u_{l,m}^{j+\frac12}\\&\quad+\frac{h^2}{2}[12\,\hat{g}_{l,m}^{j}+h^2(\hat{g}_{xx_{l,m}}^{j}+\hat{g}_{yy_{l,m}}^{j})],\quad l,m=1,\dots,N,\ j=0,\dots,J,\end{aligned} \tag{37.2}$$

respectively. The difference formulas (36.1)–(36.2) and (37.1)–(37.2) are of $O(k^2+h^4)$ and $O(k^2+kh^2+h^4)$, respectively, and free from the terms of the form $1/x_{l\pm1}$ and $1/y_{m\pm1}$, $l,m=1,\dots,N$, hence can easily be solved in the region $[0<x<1]\times[0<y<1]\times[t>0]$.

5 Stability Analysis and ADI Schemes for (P3)

Let us consider the two-dimensional nonlinear fourth-order parabolic problem (P3).

We introduce a new variable

$$w(x,y,t)=\nabla^2u(x,y,t)-u_t(x,y,t).$$

Since the value of u is known explicitly at $t=0$, this implies all its successive partial derivatives with respect to x and y are known at $t=0$, i.e. the values of $u_x,u_{xx},\dots,u_y,u_{yy},\dots$, etc. are known initially. As

$$\nabla^2u(x,y,0)=u_{xx}(x,y,0)+u_{yy}(x,y,0)=u_{0_{xx}}(x,y)+u_{0_{yy}}(x,y),$$

this implies that the value of w is known at $t=0$. Similarly, as u and u_{yy} are prescribed at $y=a$ and $y=b$, so their derivatives with respect to t and x are also known there, i.e. $u_t,u_{tt},\dots,u_x,u_{xx},\dots$, etc. are known at $y=a$ and $y=b$. Hence w is determined at the boundaries $y=a$ and $y=b$. Likewise w is known on all sides of the solution domain.

Then the time-dependent biharmonic equation (7) can be decoupled as two-heat conduction equations on the same domain:

$$\nabla^2u=\frac{\partial u}{\partial t}+w,\qquad (x,y)\in\Omega,\ t>0, \tag{38.1}$$

$$\nabla^2w=\frac{\partial w}{\partial t}+\psi(x,y,t,u),\quad (x,y)\in\Omega,\ t>0. \tag{38.2}$$

The associated initial and Dirichlet boundary conditions can be written as

$$
\begin{aligned}
u(x,y,0) &= u_0(x,y), & w(x,y,0) &= u_{0_{xx}}(x,y) + u_{0_{yy}}(x,y) - u_1(x,y), & (x,y) &\in \Omega,\\
u(x,a,t) &= a_{10}(x,t), & w(x,a,t) &= a_{10_{xx}}(x,t) + a_{20}(x,t) - a_{10_t}(x,t), & a &\le x \le b,\ t>0,\\
u(x,b,t) &= a_{11}(x,t), & w(x,b,t) &= a_{11_{xx}}(x,t) + a_{21}(x,t) - a_{11_t}(x,t), & a &\le x \le b,\ t>0,\\
u(a,y,t) &= b_{10}(y,t), & w(a,y,t) &= b_{02}(y,t) + b_{10_{yy}}(y,t) - b_{10_t}(y,t), & a &\le y \le b,\ t>0,\\
u(b,y,t) &= b_{11}(y,t), & w(b,y,t) &= b_{12}(y,t) + b_{11_{yy}}(y,t) - b_{11_t}(y,t), & a &\le y \le b,\ t>0.
\end{aligned}
$$

Let $W_{l,m}^j$ be the exact and let $w_{l,m}^j$ be the approximate solution value of $w(x,y,t)$ at the grid point (x_l, y_m, t_j). We denote

$$
\begin{aligned}
\widehat{\psi}_{l,m}^j &= \psi(x_l, y_m, t_j + \tfrac{k}{2}, \mu_t U_{l,m}^{j+\frac12}),\\
\widehat{\psi}_{l\pm1,m}^j &= \psi(x_{l\pm1}, y_m, t_j + \tfrac{k}{2}, \mu_t U_{l\pm1,m}^{j+\frac12}),\\
\widehat{\psi}_{l,m\pm1}^j &= \psi(x_l, y_{m\pm1}, t_j + \tfrac{k}{2}, \mu_t U_{l,m\pm1}^{j+\frac12}).
\end{aligned}
$$

We consider two sets of difference methods described below.

A two-level implicit method of accuracy $O(k^2+h^2)$ of Crank–Nicolson type for system (38.1)–(38.2) can be written in operator form as

$$\lambda(\delta_x^2+\delta_y^2)\mu_t U_{l,m}^{j+\frac12} = \delta_t U_{l,m}^{j+\frac12} + k\mu_t W_{l,m}^{j+\frac12} + O(k^3+kh^2), \tag{39.1}$$

$$\lambda(\delta_x^2+\delta_y^2)\mu_t W_{l,m}^{j+\frac12} = \delta_t W_{l,m}^{j+\frac12} + k\widehat{\psi}_{l,m}^j + O(k^3+kh^2), \quad l,m=1,\dots,N,\ j=0,\dots,J. \tag{39.2}$$

A two-level implicit method of accuracy $O(k^2+h^4)$ of Crandall type [5] for system (38.1)–(38.2) can be written in operator form as

$$\lambda[6(\delta_x^2+\delta_y^2)+\delta_x^2\delta_y^2]\mu_t U_{l,m}^{j+\frac12} = \frac12[12+\delta_x^2+\delta_y^2]\delta_t U_{l,m}^{j+\frac12} + \frac{k}{2}[12+\delta_x^2+\delta_y^2]\mu_t W_{l,m}^{j+\frac12} + O(k^3+kh^4), \tag{40.1}$$

$$
\begin{aligned}
\lambda[6(\delta_x^2+\delta_y^2)+\delta_x^2\delta_y^2]\mu_t W_{l,m}^{j+\frac12} &= \frac12[12+\delta_x^2+\delta_y^2]\delta_t W_{l,m}^{j+\frac12} + \frac{k}{2}[\widehat{\psi}_{l+1,m}^j + \widehat{\psi}_{l-1,m}^j + \widehat{\psi}_{l,m+1}^j + \widehat{\psi}_{l,m-1}^j + 8\widehat{\psi}_{l,m}^j]\\
&\quad + O(k^3+kh^4), \quad l,m=1,\dots,N,\ j=0,\dots,J.
\end{aligned} \tag{40.2}
$$

For stability, we consider the linear two-dimensional fourth-order parabolic equation of the form

$$\nabla^4 u - 2\Big(\frac{\partial^3 u}{\partial x^2\partial t} + \frac{\partial^3 u}{\partial y^2 \partial t}\Big) + \frac{\partial^2 u}{\partial t^2} = \psi(x,y,t), \quad (x,y)\in\Omega,\ t>0, \tag{41}$$

Applying methods (39.1)–(39.2) and (40.1)–(40.2) to the differential equation (41) results in the following block tri-diagonal matrix structure:

$$\boldsymbol{A}\boldsymbol{Z}^{j+1} = \boldsymbol{B}\boldsymbol{Z}^j + \boldsymbol{l}, \quad j=0,1,2,\dots, \tag{42}$$

where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{0} & \boldsymbol{A}_1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{B}_2 \\ \boldsymbol{0} & \boldsymbol{B}_1 \end{bmatrix}, \quad \boldsymbol{Z} = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{w} \end{bmatrix}, \quad \boldsymbol{l} = \begin{bmatrix} \boldsymbol{l}_1 \\ \boldsymbol{l}_2 \end{bmatrix}.$$

The submatrices for method (39.1)–(39.2) are given by

$$\boldsymbol{A}_1 = [\boldsymbol{P},\boldsymbol{Q},\boldsymbol{P}], \quad \boldsymbol{A}_2 = [\boldsymbol{0}, \tfrac{k}{2}\boldsymbol{I}, \boldsymbol{0}], \quad \boldsymbol{B}_1 = [\boldsymbol{R},\boldsymbol{S},\boldsymbol{R}], \quad \boldsymbol{B}_2 = [\boldsymbol{0}, -\tfrac{k}{2}\boldsymbol{I}, \boldsymbol{0}],$$

where

$$\boldsymbol{P} = -\frac{\lambda}{2}[0,1,0], \quad \boldsymbol{Q} = [0,1,0] - \frac{\lambda}{2}[1,-4,1], \quad \boldsymbol{R} = \frac{\lambda}{2}[0,1,0], \quad \boldsymbol{S} = [0,1,0] + \frac{\lambda}{2}[1,-4,1]$$

and the submatrices for method (40.1)–(40.2) are given by

$$\boldsymbol{A}_1 = [\boldsymbol{P},\boldsymbol{Q},\boldsymbol{P}], \quad \boldsymbol{A}_2 = [\boldsymbol{C},\boldsymbol{D},\boldsymbol{C}], \quad \boldsymbol{B}_1 = [\boldsymbol{R},\boldsymbol{S},\boldsymbol{R}], \quad \boldsymbol{B}_2 = [\boldsymbol{E},\boldsymbol{F},\boldsymbol{E}],$$

where

$$\boldsymbol{P} = [0,1,0] - \lambda[1,4,1], \quad \boldsymbol{Q} = [1,8,1] - 4\lambda[1,-5,1], \quad \boldsymbol{C} = \frac{k}{2}[0,1,0], \quad \boldsymbol{D} = \frac{k}{2}[1,8,1],$$
$$\boldsymbol{R} = [0,1,0] + \lambda[1,4,1], \quad \boldsymbol{S} = [1,8,1] + 4\lambda[1,-5,1], \quad \boldsymbol{E} = -\frac{k}{2}[0,1,0], \quad \boldsymbol{F} = -\frac{k}{2}[1,8,1].$$

Here we denote

$$[a,b,c] = \begin{bmatrix} b & c & 0 & \cdots & 0 \\ a & b & c & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & a & b & c \\ 0 & \cdots & 0 & a & b \end{bmatrix}$$

as the $N \times N$ tri-diagonal matrix and its eigenvalues are given by

$$v_s = b + 2\sqrt{a\,c}\cos(2\phi), \quad 2\,\phi = \frac{s\pi}{N+1}, \quad s = 1,\dots,N.$$

Also, $\boldsymbol{u} = (u_1, u_2, \dots, u_N)^T$ and $\boldsymbol{w} = (w_1, w_2, \dots, w_N)^T$ are solution vectors; l_1 and l_2 are right-hand side column vectors of order N that consist of homogenous functions, initial and boundary values of the block system (42).

For discussing stability, we consider the homogenous part of (42), which may be written as

$$\boldsymbol{Z}^{j+1} = (\boldsymbol{A}^{-1}\boldsymbol{B})\boldsymbol{Z}^j, \quad j = 0,1,2,\dots.$$

Let $\boldsymbol{\varepsilon}^j = \boldsymbol{z}^j - \boldsymbol{Z}^j$ be the error vector at the jth iterate (in the absence of round-off errors), where

$$\boldsymbol{Z}^j = \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{W} \end{bmatrix}^j, \quad \boldsymbol{Z}^{j+1} = \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{W} \end{bmatrix}^{j+1}.$$

The error equation can be written as

$$\boldsymbol{\varepsilon}^{j+1} = \boldsymbol{H}\boldsymbol{\varepsilon}^j, \quad j = 0,1,2,\dots,$$

where $\boldsymbol{H} = \boldsymbol{A}^{-1}\boldsymbol{B}$ is the amplification matrix.

The eigenvalues of $\boldsymbol{A_1}, \boldsymbol{A_2}, \boldsymbol{B_1}$ and $\boldsymbol{B_2}$ for method (39.1)–(39.2) are given by $1 + 2\lambda(\sin^2\phi_1 + \sin^2\phi_2)$, $\frac{k}{2}$, $1 - 2\lambda(\sin^2\phi_1 + \sin^2\phi_2)$ and $\frac{-k}{2}$, respectively, where $\phi_1 = (\pi s)/(N+1)$, $s = 1, 2, \dots, N$, and $\phi_2 = (\pi t)/(N+1)$, $t = 1, 2, \dots, N$. Thus, the eigenvalues of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ for the difference method (39.1)–(39.2) are given by $1 + 2\lambda(\sin^2\phi_1 + \sin^2\phi_2)$ and $1 - 2\lambda(\sin^2\phi_1 + \sin^2\phi_2)$, respectively.

The eigenvalues of $\boldsymbol{A_1}, \boldsymbol{A_2}, \boldsymbol{B_1}$ and $\boldsymbol{B_2}$ for method (40.1)–(40.2) are given by

$$12 - 4(\sin^2\phi_1 + \sin^2\phi_2) + 8\lambda(\sin^2\phi_1 + 2\sin^2\phi_1\cos^2\phi_2 + 3\sin^2\phi_2), \quad \frac{k}{2}[12 - 4(\sin^2\phi_1 + \sin^2\phi_2)],$$
$$12 - 4(\sin^2\phi_1 + \sin^2\phi_2) - 8\lambda(\sin^2\phi_1 + 2\sin^2\phi_1\cos^2\phi_2 + 3\sin^2\phi_2), \quad -\frac{k}{2}[12 - 4(\sin^2\phi_1 + \sin^2\phi_2)],$$

respectively. Hence, the eigenvalues of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ for the difference method (40.1)–(40.2) are given by

$$12 - 4(\sin^2\phi_1 + \sin^2\phi_2) + 8\lambda(\sin^2\phi_1 + 2\sin^2\phi_1\cos^2\phi_2 + 3\sin^2\phi_2)$$

and

$$12 - 4(\sin^2\phi_1 + \sin^2\phi_2) - 8\lambda(\sin^2\phi_1 + 2\sin^2\phi_1\cos^2\phi_2 + 3\sin^2\phi_2),$$

respectively.

Since the matrices $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}$ commute each other, the eigenvalues $\xi_{s\,t}$ of $\boldsymbol{A}^{-1}\boldsymbol{B}$ for method (39.1)–(39.2) are given by

$$\xi_{s\,t} = \frac{1 - 2\lambda(\sin^2\phi_1 + \sin^2\phi_2)}{1 + 2\lambda(\sin^2\phi_1 + \sin^2\phi_2)},$$

and for method (40.1)–(40.2),

$$\xi_{st} = \frac{12 - 4(\sin^2\phi_1 + \sin^2\phi_2) - 8\lambda(\sin^2\phi_1 + 2\sin^2\phi_1\cos^2\phi_2 + 3\sin^2\phi_2)}{12 - 4(\sin^2\phi_1 + \sin^2\phi_2) + 8\lambda(\sin^2\phi_1 + 2\sin^2\phi_1\cos^2\phi_2 + 3\sin^2\phi_2)}.$$

Since $0 \le \sin^2\phi_1, \sin^2\phi_2 \le 1$ and $\lambda > 0$, the condition $|\xi_{st}| \le 1$ is always satisfied and hence methods (39.1)–(39.2) and (40.1)–(40.2) are unconditionally stable.

The difference methods (39.1)–(39.2) and (40.1)–(40.2) results in a system of equations with large bandwidth at each time level. We can write (39.1)–(39.2) and (40.1)–(40.2) in product form as

$$\Big[1 - \frac{\lambda}{2}\delta_x^2\Big]\Big[1 - \frac{\lambda}{2}\delta_y^2\Big]u_{l,m}^{j+1} = \Big[1 + \frac{\lambda}{2}\delta_x^2\Big]\Big[1 + \frac{\lambda}{2}\delta_y^2\Big]u_{l,m}^{j} - \frac{k}{2}(w_{l,m}^{j+1} + w_{l,m}^{j}) \equiv R_u, \tag{43.1}$$

$$\Big[1 - \frac{\lambda}{2}\delta_x^2\Big]\Big[1 - \frac{\lambda}{2}\delta_y^2\Big]w_{l,m}^{j+1} = \Big[1 + \frac{\lambda}{2}\delta_x^2\Big]\Big[1 + \frac{\lambda}{2}\delta_y^2\Big]w_{l,m}^{j} - k\,\widehat{\psi}_{l,m}^{j} \equiv R_w, \quad l, m = 1, \dots, N,\ j = 0, \dots, J, \tag{43.2}$$

and

$$\Big[1 + \frac{1}{12}(1 - 6\lambda)\delta_x^2\Big]\Big[1 + \frac{1}{12}(1 - 6\lambda)\delta_y^2\Big]u_{l,m}^{j+1}$$
$$= \Big[1 + \frac{1}{12}(1 + 6\lambda)\delta_x^2\Big]\Big[1 + \frac{1}{12}(1 + 6\lambda)\delta_y^2\Big]u_{l,m}^{j} - \frac{k}{12}\Big(1 + \frac{\delta_x^2}{12} + \frac{\delta_y^2}{12}\Big)(w_{l,m}^{j+1} + w_{l,m}^{j}) \equiv RR_u, \tag{44.1}$$

$$\Big[1 + \frac{1}{12}(1 - 6\lambda)\delta_x^2\Big]\Big[1 + \frac{1}{12}(1 - 6\lambda)\delta_y^2\Big]w_{l,m}^{j+1}$$
$$= \Big[1 + \frac{1}{12}(1 + 6\lambda)\delta_x^2\Big]\Big[1 + \frac{1}{12}(1 + 6\lambda)\delta_y^2\Big]w_{l,m}^{j} - \frac{k}{12}\sum\psi \equiv RR_w, \quad l, m = 1, \dots, N,\ j = 0, \dots, J, \tag{44.2}$$

respectively, where

$$\sum\psi = \widehat{\psi}_{l+1,m}^{j} + \widehat{\psi}_{l-1,m}^{j} + \widehat{\psi}_{l,m+1}^{j} + \widehat{\psi}_{l,m-1}^{j} + 8\widehat{\psi}_{l,m}^{j}.$$

The additional terms added in (43.1)–(43.2) and (44.1)–(44.2) are of high order and do not affect the accuracy of the schemes. For instance, we have added the extra terms $(\frac{1}{144} + \frac{\lambda^2}{4})\delta_x^2\delta_y^2 u_{l,m}^{j+1}$ and $(\frac{1}{144} + \frac{\lambda^2}{4})\delta_x^2\delta_y^2 u_{l,m}^{j}$ on the left- and right-hand sides of (44.1), respectively. Expanding

$$u_{l,m}^{j+1} = u_{l,m}^{j} + O(k),$$

we observe that the overall order of the extra terms appearing in (44.1) is

$$\Big(\frac{1}{144} + \frac{\lambda^2}{4}\Big)\delta_x^2\delta_y^2(u_{l,m}^{j+1} - u_{l,m}^{j}) = O(k^3 + kh^4).$$

Similarly, the extra terms appearing in (43.1)–(43.2) does not affect the order of the method.

Now, in order to facilitate the computation, we may write (43.1)–(43.2) and (44.1)–(44.2) in two-step ADI methods (see [1, 9–11]) as

$$\Big[1 - \frac{\lambda}{2}\delta_y^2\Big]u_{l,m}^{*\,j+1} = R_u, \tag{45.1}$$

$$\Big[1 - \frac{\lambda}{2}\delta_x^2\Big]u_{l,m}^{j+1} = u_{l,m}^{*\,j+1}, \tag{45.2}$$

$$\Big[1 - \frac{\lambda}{2}\delta_y^2\Big]w_{l,m}^{*\,j+1} = R_w, \tag{45.3}$$

$$\Big[1 - \frac{\lambda}{2}\delta_x^2\Big]w_{l,m}^{j+1} = w_{l,m}^{*\,j+1} \tag{45.4}$$

and

$$\Big[1 + \frac{1}{12}(1 - 6\lambda)\delta_y^2\Big]u_{l,m}^{**\,j+1} = RR_u, \tag{46.1}$$

$$\Big[1 + \frac{1}{12}(1 - 6\lambda)\delta_x^2\Big]u_{l,m}^{j+1} = u_{l,m}^{**\,j+1}, \tag{46.2}$$

$$\Big[1 + \frac{1}{12}(1 - 6\lambda)\delta_y^2\Big]w_{l,m}^{**\,j+1} = RR_w, \tag{46.3}$$

$$\Big[1 + \frac{1}{12}(1 - 6\lambda)\delta_x^2\Big]w_{l,m}^{j+1} = w_{l,m}^{**\,j+1}, \tag{46.4}$$

respectively, where $u_{l,m}^{*\,j+1}$, $w_{l,m}^{*\,j+1}$, $u_{l,m}^{**\,j+1}$ and $w_{l,m}^{**\,j+1}$ are intermediate values. The intermediate boundary conditions needed for the solution of these intermediate values can be determined from (45.2), (45.4), (46.2) and (46.4), respectively. For instance, consider (45.1):

$$\left[1 - \frac{\lambda}{2}\delta_y^2\right]u_{l,m}^{*\,j+1} = R_u,$$

or, equivalently,

$$-\frac{\lambda}{2}u_{l,m-1}^{*\,j+1} + (1+\lambda)u_{l,m}^{*\,j+1} - \frac{\lambda}{2}u_{l,m+1}^{*\,j+1} = R_u, \quad l, m = 1, \dots, N.$$

For $m = 1, N$, the intermediate boundary values $u_{l,0}^{*\,j+1}$ and $u_{l,N+1}^{*\,j+1}$ for $l = 1, \dots, N$ are obtained using (45.2) in the following manner:

$$u_{l,0}^{*\,j+1} = -\frac{\lambda}{2}u_{l-1,0}^{j+1} + (1+\lambda)u_{l,0}^{j+1} - \frac{\lambda}{2}u_{l+1,0}^{j+1}, \quad l = 1, \dots, N. \tag{47}$$

Using (47), the value of $u_{l,0}^{*\,j+1}$ is determined since u is known at the boundary. Similarly, $u_{l,N+1}^{*\,j+1}$ is also determined using (45.2) and the boundary values of u.

Note that the left-hand sides of (45.1)–(45.2) are factorizations into y- and x-differences, respectively, which enables us to solve by sweeping first (45.1) in the y- and then (45.2) in the x-direction and these sweeps require only the solution of tri-diagonal systems. Likewise, the matrices on the left-hand sides of (45.3)–(45.4) and (46.1)–(46.4) are tri-diagonal, therefore, easily solvable using tri-diagonal solvers.

6 Computational Results

In order to illustrate the high order accuracy of the methods, we performed some numerical experiments to compare the numerical solutions with the exact solutions. For evaluating the solution of (P1), we have chosen $\theta = 0.5$. Problem (P3) have been solved using the ADI methods (45.1)–(45.4) and (46.1)–(46.4). For problems (P2) and (P3), we have two unknowns u and $\nabla^2 u \equiv v$ at each interior grid point and the resulting matrix systems are solved using appropriate block iterative methods. The nonlinear equations have been solved using Newton's nonlinear block Gauss–Seidel iteration method (see Hageman and Young [13], Kelly [15] and Saad [28]) whereas the block Gauss–Seidel iteration method has been used for solving linear equations of these kind. The initial vector $(0, 0)$ is used in all the cases and the iterations were terminated when the absolute error tolerance $|u^{(k+1)} - u^{(k)}| + |v^{(k+1)} - v^{(k)}| \le 2 \times 10^{-12}$ was achieved. All computations were carried out using MATLAB and satisfactory results are obtained.

In Tables 1–8, we report maximum absolute errors (L_∞-norm) of the computed solutions which are defined as, say for the solution $u(x, y, t)$,

$$\text{MAE} = \max_{1\le l,m\le N} |u(x_l, y_m, T) - U(x_l, y_m, T)|,$$

for different grid sizes, where $U(x_l, y_m, T)$ is the exact solution and $u(x_l, y_m, T)$ is the corresponding approximate solution at the grid point (x_l, y_m) obtained by proposed methods at time T. Since we are interested in the fourth-order accuracy in space, we have to restrict the time step to $k = \lambda h^2$, for the fixed value of the parameter $\lambda = 1.6$. For this choice, our method behaves like a fourth-order method in space which is verified using the formula

$$\text{Order} = \frac{\log(E_h) - \log(E_{\frac{h}{2}})}{\log(2)},$$

where E_h and $E_{\frac{h}{2}}$ are maximum absolute errors corresponding to the mesh lengths h and $\frac{h}{2}$, respectively.

Note that the suggested method (15.1)–(15.2) for the fourth-order parabolic problem (P1) is a three-level scheme. The value of u and v at $t = 0$ is known from the initial conditions and using successive derivative values. To start any computation, it is essential to know the numerical value of u and v of required accuracy at the first time level, $t = k$. We consider an explicit scheme of $O(k^2)$ for the solution variables u and v at $t = k$.

Since the values of u and u_t are specified at $t = 0$, this means all their successive partial derivatives with respect to x and y are known at $t = 0$, i.e. the values $u_x, u_{xx}, u_{xxx}, u_{xxxx}, \dots, u_y, u_{yy}, u_{yyy}, u_{yyyy}, \dots, u_{xxy}, u_{xxyy}, \dots, u_{tx}, u_{ty}, \dots$, etc. are known at $t = 0$. This means the values of v and $\nabla^4 u = u_{xxxx} + 2u_{xxyy} + u_{yyyy}$ are known initially.

Approximations for u and v of $O(k^2)$ at $t = k$ may be written as

$$u^1_{l,m} = u^0_{l,m} + ku^0_{t_{l,m}} + \frac{k^2}{2} u^0_{tt_{l,m}} + O(k^3), \tag{48}$$

$$v^1_{l,m} = v^0_{l,m} + kv^0_{t_{l,m}} + \frac{k^2}{2} v^0_{tt_{l,m}} + O(k^3). \tag{49}$$

From (1), we have

$$\frac{\partial^2 u}{\partial t^2} = -A(x, y, t, u, \nabla^2 u)\nabla^4 u + f(x, y, t, u, u_t, u_x, u_y, \nabla^2 u, \nabla^2 u_x, \nabla^2 u_y). \tag{50}$$

Differentiating (50) twice successively with respect to x and y and then adding, we obtain

$$\frac{\partial^2 v}{\partial t^2} = \nabla^2[-A(x, y, t, u, \nabla^2 u)\nabla^4 u + f(x, y, t, u, u_t, u_x, u_y, \nabla^2 u, \nabla^2 u_x, \nabla^2 u_y)]. \tag{51}$$

Thus using the initial and successive derivative values, from (50) and (51), we can obtain the values of $u^0_{tt_l}$ and $v^0_{tt_l}$, respectively, and then finally, from (48) and (49) we can compute the values of u and v at first time level, i.e. at $t = k$.

Example 1 (Dynamic Thin Plate Equation). We consider the problem of determining the transverse vibrations of a square homogenous thin plate hinged at its boundaries [4, 9]. The initial boundary value problem may be written as

$$\nabla^4 u + \frac{\partial^2 u}{\partial t^2} = f(x, y, t), \quad 0 < x, y < 1, \ t > 0,$$

where $f(x, y, t) = (4\pi^4 + 1) \sinh t \sin(\pi x) \sin(\pi y)$, subject to the initial conditions

$$u(x, y, 0) = 0, \quad u_t(x, y, 0) = \sin(\pi x) \sin(\pi y), \quad 0 < x, y < 1,$$

and the boundary conditions

$$\begin{aligned} &u(x, 0, t) = 0, \quad u_{yy}(x, 0, t) = 0, \quad u(x, 1, t) = 0, \quad u_{yy}(x, 1, t) = 0, \quad 0 \le x \le 1, \ t > 0, \\ &u(0, y, t) = 0, \quad u_{xx}(0, y, t) = 0, \quad u(1, y, t) = 0, \quad u_{xx}(1, y, t) = 0, \quad 0 \le y \le 1, \ t > 0. \end{aligned}$$

The exact solution is

$$u(x, y, t) = \sinh t \sin(\pi x) \sin(\pi y).$$

The maximum absolute errors in the solution variable u, the moment $\nabla^2 u$ and order of convergence computed by difference methods (15.1)–(15.2) and (11.1)–(11.2) are listed in Table 1 for fixed $\lambda = 1.6$ at $t = 1$.

h		**$O(k^2 + h^4)$-method (15.1)–(15.2)**		**$O(k^2 + h^2)$-method (11.1)–(11.2)**	
		MAE	**Order**	**MAE**	**Order**
1/8	u	5.2066(−04)	–	3.0546(−02)	–
	$\nabla^2 u$	8.7601(−03)	–	2.9867(−01)	–
1/16	u	3.1521(−05)	4.0460	7.3086(−03)	2.0633
	$\nabla^2 u$	5.2665(−04)	4.0560	6.9370(−02)	2.1062
1/32	u	1.9681(−06)	4.0014	1.8211(−03)	2.0048
	$\nabla^2 u$	3.2867(−05)	4.0021	1.7292(−02)	2.0042

Table 1. Maximum absolute errors and order of convergence for Example 1 at $t = 1.0$.

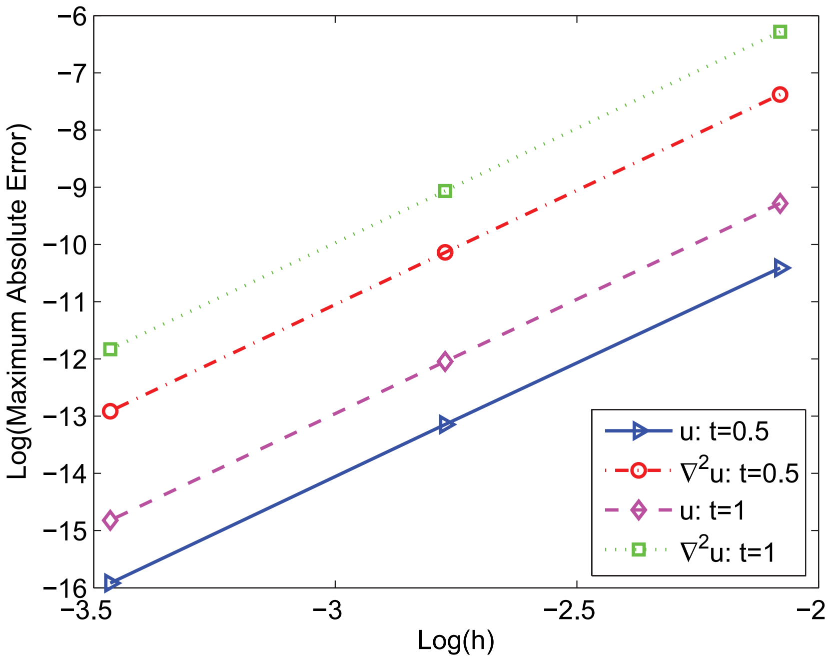

Figure 1. Order of convergence for Example 2 at $t = 0.5$ and $t = 1$ by implicit method (15.1)–(15.2).

Example 2 (Two-DImensional Boussinesq-Type Equation). We consider the Boussinesq water equation in the form

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u + \nabla^2 u^2 - \nabla^4 u + f(x, y, t), \quad 0 < x, y < 1, \; t > 0, \tag{52}$$

where the function $f(x, y, t) = [1 - 8e^{x+y} \sin t]e^{x+y} \sin t$ is chosen so that

$$u(x, y, t) = e^{x+y} \sin t$$

solves (52) exactly and equipped with the initial and boundary conditions of the form (2.1)–(2.2) and (3.1)–(3.4). This equation was the first model for the propagation of surface waves over shallow inviscid fluid layer [3, 32]. The two-dimensional Boussinesq model is still less accessible analytically which requires developing numerical techniques. Table 2 gives L_∞-discrete errors and order of convergence by the difference methods (15.1)–(15.2) and (11.1)–(11.2) for different values of h with $\lambda = 1.6$ at $t = 1$. In Figure 1, we display in a Log/Log scale the maximum absolute errors in $u(x, y, t)$ and $\nabla^2 u(x, y, t)$ using (15.1)–(15.2) at $t = 0.5$ and $t = 1$. It is clear from Figure 1 that the slope of the graph is almost constant which is around four.

h		$O(k^2 + h^4)$-method (15.1)–(15.2)		$O(k^2 + h^2)$-method (11.1)–(11.2)	
		MAE	**Order**	**MAE**	**Order**
1/8	u	9.3031(−05)	–	7.0599(−04)	–
	$\nabla^2 u$	1.8644(−03)	–	5.0558(−03)	–
1/16	u	5.8690(−06)	3.9865	2.0274(−04)	1.8000
	$\nabla^2 u$	1.1565(−04)	4.0109	1.6523(−03)	1.6135
1/32	u	3.6667(−07)	4.0006	5.1966(−05)	1.9640
	$\nabla^2 u$	7.2656(−06)	3.9925	4.3685(−04)	1.9193

Table 2. Maximum absolute errors and order of convergence for Example 2 at $t = 1.0$.

h		$O(k^2+h^4)$-method (15.1)–(15.2)		$O(k^2+h^2)$-method (11.1)–(11.2)	
		MAE	Order	MAE	Order
1/8	u	3.4135(−05)	–	3.1358(−03)	–
	$\nabla^2 u$	1.5835(−03)	–	4.9241(−02)	–
1/16	u	2.0913(−06)	4.0288	8.2649(−04)	1.9238
	$\nabla^2 u$	1.0394(−04)	3.9293	1.2996(−02)	1.9218
1/32	u	1.3766(−07)	3.9252	2.1034(−04)	1.9743
	$\nabla^2 u$	6.6455(−06)	3.9672	3.3844(−03)	1.9411

Table 3. Maximum absolute errors and order of convergence for Example 3 at $t = 1.0$.

Example 3. We consider a fourth-order parabolic equation of the form

$$\frac{\partial^2 u}{\partial t^2} + \nabla^4 u - 2\nabla^2 u + 3u = f(x,y,t), \quad 0 < x, y < 1,\ t > 0, \tag{53}$$

where $f(x,y,t) = 4(\pi^4 + \pi^2 + 1)\sinh t \cos(\pi x)\cos(\pi y)$, subject to the initial conditions

$$u(x,y,0) = 0, \quad u_t(x,y,0) = \cos(\pi x)\cos(\pi y), \quad 0 < x, y < 1,$$

and the boundary conditions

$$\begin{aligned}
u(x,0,t) &= \sinh t\cos(\pi x), & u_{yy}(x,0,t) &= -\pi^2 \sinh t\cos(\pi x), & 0 \le x \le 1,\ t > 0,\\
u(x,1,t) &= -\sinh t\cos(\pi x), & u_{yy}(x,1,t) &= \pi^2 \sinh t\cos(\pi x), & 0 \le x \le 1,\ t > 0,\\
u(0,y,t) &= \sinh t\cos(\pi y), & u_{xx}(0,y,t) &= -\pi^2 \sinh t\cos(\pi y), & 0 \le y \le 1,\ t > 0,\\
u(1,y,t) &= -\sinh t\cos(\pi y), & u_{xx}(1,y,t) &= \pi^2 \sinh t\cos(\pi y), & 0 \le y \le 1,\ t > 0.
\end{aligned}$$

It is easy to verify that

$$u(x,y,t) = \sinh t\cos(\pi x)\cos(\pi y)$$

is the exact solution of (53). The computations are performed from $t = 0$ until the final time $T = 1.0$. The maximum absolute errors and order of convergence for u and the moment $\nabla^2 u$ obtained by employing difference methods (15.1)–(15.2) and (11.1)–(11.2) are presented in Table 3 for different values of h with $\lambda = 1.6$ at $t = 1$.

Example 4 (Extended Fisher–Kolmogorov Equation, [16]). Consider the problem for the EFK equation (5) in $\Omega \times (0,T]$, where $\Omega = (0,1) \times (0,1)$, with the initial condition

$$u(x,y,0) = x^3(1-x)^3 y^3(1-y)^3, \quad (x,y) \in \Omega, \tag{54}$$

and the boundary conditions

$$u = 0, \quad \nabla^2 u = 0, \quad (x,y,t) \in \partial\Omega \times (0,T]. \tag{55}$$

The profile of the approximate solution of equation (5) with $y = 0.1$ subject to (54)–(55) obtained using the difference method (29.1)–(29.2) is presented in Figure 2 with $J = 40$ and $N + 1 = 40$ at $t = 1$. It is observed that the solution profile shows same characteristics as in [16].

Example 5 (Nonhomogeneous Extended Fisher–Kolmogorov Equation, [16]). Consider the following nonlinear nonhomogeneous EFK equation:

$$\frac{\partial u}{\partial t} + \gamma\nabla^4 u - \nabla^2 u + f(u) = g(x,y,t), \quad (x,y) \in \Omega = (0,1) \times (0,1),\ t > 0, \tag{56}$$

where $f(u) = u^3 - u$. The initial condition associated with (56) is

$$u(x,y,0) = \sin(2\pi x)\sin(2\pi y), \quad (x,y) \in \Omega, \tag{57}$$

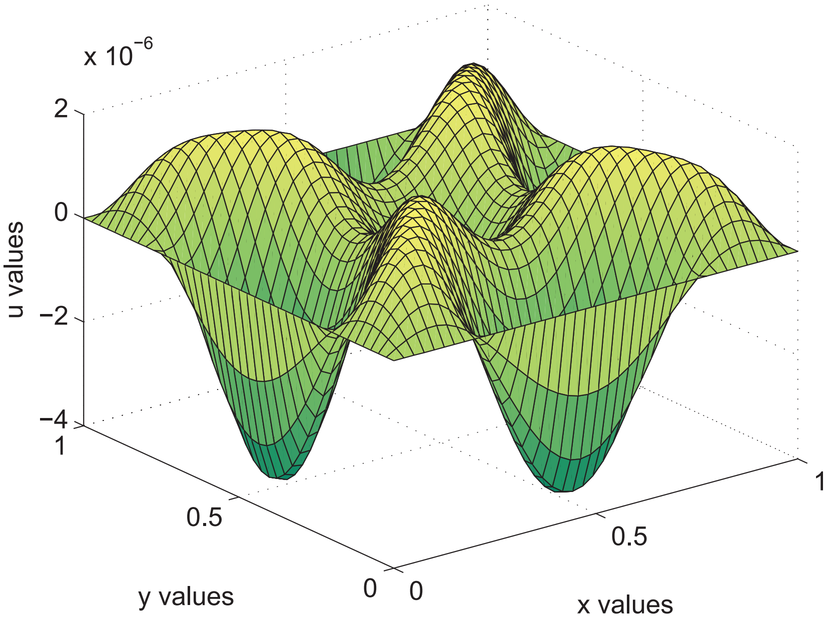

Figure 2. The profile of numerical solution for Example 4 with $\gamma = 0.1$, $J = 40$ and $N + 1 = 40$ at $t = 1$.

and the boundary conditions are given by

$$u = 0, \quad \nabla^2 u = 0, \quad (x, y) \in \partial\Omega, \ t > 0. \tag{58}$$

Here $g(x, y, t) = (\sin(2\pi x)^2 \sin(2\pi y)^2 e^{-t^2} - 2 + 64\gamma\pi^4 + 8\pi^2)\sin(2\pi x)\sin(2\pi y)e^{-t}$ and

$$u(x, y, t) = \sin(2\pi x)\sin(2\pi y)e^{-t}$$

is the exact solution. We compute the numerical solution of problem (56)–(58) for $\gamma = 0.01$ using finite difference method (29.1)–(29.2) by choosing the time step size $k = h$ for each corresponding spatial mesh length h and report the results in Table 4. With this choice of time step size, the theoretical order of convergence of the $O(k^2 + kh^2 + h^4)$-method (29.1)–(29.2) becomes $O(h^2)$, i.e. the method is second-order accurate in space and we have compared our results with the second-order method discussed by Khiari and Omrani [16]. From Table 4, it is clear that our results are better than that of [16]. Also, for a fixed parameter value $\lambda = 1.6$, the proposed method (29.1)–(29.2) behaves like a fourth-order method which the computed results in Table 5 exhibit for various values of γ. For a fixed value of λ, we found from Table 5 that the order of convergence of the proposed method (29.1)–(29.2) is indeed four.

		Khiari and Omrani [16]		Present method (29.1)–(29.2)	
J	$N+1$	MAE	Order	MAE	Order
15	15	0.0064	–	2.4894(–04)	–
20	20	0.0038	–	1.2984(–04)	–
25	25	0.0022	–	7.9560(–05)	–
30	30	0.0016	2.001	5.3588(–05)	2.2158
40	40	8.1310(–04)	2.2245	2.9780(–05)	2.1243
50	50	4.4541(–04)	2.3043	1.8778(–05)	2.0830

Table 4. Maximum absolute errors and order of convergence for Example 5 at $t = 1.0$ for different values of $k = \frac{1}{J}$ and $h = \frac{1}{N+1}$ with $\gamma = 0.01$.

h		$\gamma = 0.1$		$\gamma = 0.01$		$\gamma = 0.001$	
		MAE	Order	MAE	Order	MAE	Order
1/8	u	7.3353(−04)	–	5.7259(−04)	–	4.3719(−04)	–
	$\nabla^2 u$	2.8520(−02)	–	1.5785(−02)	–	5.0585(−03)	–
1/16	u	4.7348(−05)	3.9535	3.6933(−05)	3.9545	2.8183(−05)	3.9554
	$\nabla^2 u$	1.8378(−03)	3.9559	1.0146(−03)	3.9596	3.2194(−04)	3.9738
1/32	u	2.9823(−06)	3.9888	2.3246(−06)	3.9899	1.7734(−06)	3.9902
	$\nabla^2 u$	1.1564(−04)	3.9903	6.3815(−05)	3.9909	2.0203(−05)	3.9942

Table 5. Maximum absolute errors and order of convergence for difference method (29.1)–(29.2) to Example 5 for different values of γ at $t = 1.0$ with fixed $\lambda = (k/h^2) = 1.6$.

Example 6 (Nonhomogeneous Kuramoto–Sivashinsky Equation). We consider the two-dimensional nonhomogeneous KS equation in its space derivative form [17] written as

$$\frac{\partial u}{\partial t} + \nabla^4 u + \nabla^2 u + u\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) = g(x, y, t), \quad (x, y) \in \Omega = (-1, 1) \times (-1, 1),\ t > 0,$$

where

$$g(x, y, t) = [4\pi^4 - 1 - 2\pi^2 - \pi e^{-t}(\sin(\pi x)\cos(\pi y) + \cos(\pi x)\sin(\pi y))]e^{-t}\cos(\pi x)\cos(\pi y),$$

subject to the initial conditions

$$u(x, y, 0) = \cos(\pi x)\cos(\pi y), \quad u_t(x, y, 0) = -\cos(\pi x)\cos(\pi y), \quad -1 < x, y < 1,$$

and the boundary conditions

$$\begin{aligned} u(x, -1, t) = u(x, 1, t) = -e^{-t}\cos(\pi x), &\quad u_{yy}(x, -1, t) = u_{yy}(x, 1, t) = \pi^2 e^{-t}\cos(\pi x), \quad -1 \le x \le 1, \\ u(-1, y, t) = u(1, y, t) = -e^{-t}\cos(\pi y), &\quad u_{xx}(-1, y, t) = u_{xx}(1, y, t) = \pi^2 e^{-t}\cos(\pi y), \quad -1 \le y \le 1, \end{aligned}$$

for $t > 0$. The exact solution is

$$u(x, y, t) = e^{-t}\cos(\pi x)\cos(\pi y).$$

The maximum absolute errors and order of convergence using the difference methods (29.1)–(29.2) and (24.1)–(24.2) are tabulated in Table 6 at $t = 1$ for $\lambda = 1.6$.

h		$O(k^2 + kh^2 + h^4)$-method (29.1)–(29.2)		$O(k^2 + h^2)$-method (24.1)–(24.2)	
		MAE	Order	MAE	Order
1/8	u	2.5739(−04)	–	2.7747(−02)	–
	$\nabla^2 u$	3.0307(−03)	–	4.1606(−01)	–
1/16	u	1.6040(−05)	4.0042	6.7965(−03)	2.0295
	$\nabla^2 u$	1.8480(−04)	4.0356	1.0150(−02)	2.0353

Table 6. Maximum absolute errors and order of convergence for Example 6 at $t = 1.0$.

Example 7 (Two-Dimensional Unsteady Singular Biharmonic Equation). We solve numerically the following two-dimensional fourth-order problem with singular coefficient:

$$\nabla^4 u + \frac{\partial u}{\partial t} + \frac{1}{x}u_x = g(x, y, t), \quad 0 < x, y < 1,\ t > 0, \tag{59}$$

equipped with the initial and boundary conditions of the form (2.1) and (3.1)–(3.4), where

$$g(x, y, t) = (\pi^4 - 1)e^{-t}x^2\sin(\pi y) + 2e^{-t}\sin(\pi y)(1 - 2\pi^2),$$

and the exact solution is

$$u(x, y, t) = e^{-t}x^2\sin(\pi y).$$

h		$O(k^2 + kh^2 + h^4)$-method (37.1)–(37.2)		$O(k^2 + h^2)$-method (24.1)–(24.2)	
		MAE	Order	MAE	Order
1/8	u	3.2132(−05)	–	1.8219(−03)	–
	$\nabla^2 u$	1.7244(−04)	–	6.6540(−03)	–
1/16	u	2.0018(−06)	4.0046	4.7659(−04)	1.9346
	$\nabla^2 u$	1.0211(−05)	4.0779	2.1621(−03)	1.6218
1/32	u	1.2157(−07)	4.0414	1.1933(−04)	1.9978
	$\nabla^2 u$	6.4075(−07)	3.9942	5.2892(−04)	2.0313

Table 7. Maximum absolute errors and order of convergence for Example 7 at $t = 0.5$.

This problem is singular at $x = 0$. The maximum absolute errors and order of convergence using the difference schemes (37.1)–(37.2) and (24.1)–(24.2) are tabulated in Table 7 at $t = 0.5$. It is observed from Table 7 that the order and accuracy of the scheme (37.1)–(37.2) is retained by using the approximations (35.1)–(35.8).

Example 8. We conducted an accuracy test using ADI methods (45.1)–(45.4) and (46.1)–(46.4) for the two-dimensional fourth-order linear parabolic equation (41) of the class (P3) whose exact solution is

$$u(x, y, t) = e^{-t} \sinh x \sinh y$$

and $\psi(x, y, t) = 9e^{-t} \sinh x \sinh y$. The maximum absolute errors and order of convergence for the solution variables $u(x, y, t)$ and its time-dependent Laplacian $(\nabla^2 - \frac{\partial}{\partial t})u(x, y, t)$ are reported in Table 8 at $t = 1$. In Figure 3, we plot log(L_∞-errors) against log(h) at $t = 1$ obtained using the ADI schemes (45.1)–(45.4) and (46.1)–(46.4). It is clear from Figure 3 that the slope of the graph is almost constant which is around four for the scheme (46.1)–(46.4).

h		$O(k^2 + h^4)$-ADI scheme (46.1)–(46.4)		$O(k^2 + h^2)$-ADI scheme (45.1)–(45.4)	
		MAE	Order	MAE	Order
1/8	u	5.7923(−07)	–	2.1039(−05)	–
	$\nabla^2 u - u_t$	9.6422(−06)	–	8.1786(−05)	–
1/16	u	3.6304(−08)	3.9959	5.2419(−06)	2.0049
	$\nabla^2 u - u_t$	6.0210(−07)	4.0013	1.8883(−05)	2.1148
1/32	u	2.2904(−09)	3.9865	1.3190(−06)	1.9906
	$\nabla^2 u - u_t$	3.7630(−08)	4.0000	4.6226(−06)	2.0303
1/64	u	1.4315(−10)	4.0000	3.2965(−07)	2.0004
	$\nabla^2 u - u_t$	2.3567(−09)	3.9970	1.1519(−06)	2.0047

Table 8. Maximum absolute errors and order of convergence for Example 8 at $t = 1.0$.

7 Conclusions

In this article, we have presented new three-level and two-level implicit difference methods for the solution of time-dependent biharmonic problems using coupled form. These methods are second-order accurate in time and fourth-order accurate in space and involves only nine point compact stencil at each time level without discretizing the derivative boundary conditions. Ease of implementation, avoidance of fictitious nodes outside the solution domain, compact stencil and high accuracy with smaller grid size are the major advantages of the present approach. The numerical solution of spatial Laplacian of $u(x, y, t)$, which is quite often of interest in several applied problems, is obtained as a by-product of the methods. The computed results exhibit that the methods provide fourth-order convergent results for a fixed value of mesh ratio parameter. The fourth-order

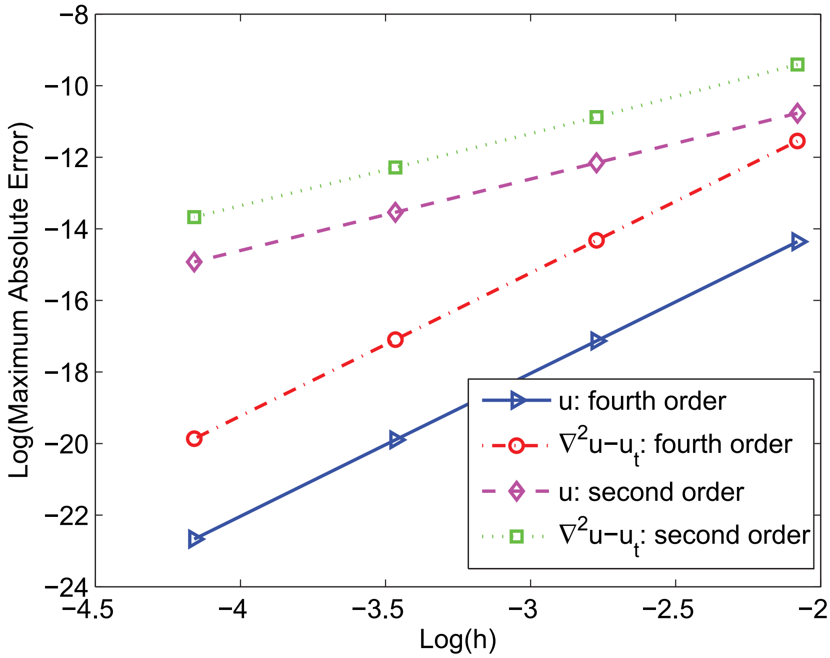

Figure 3. Order of convergence for Example 8 at $t = 1$ by ADI methods (46.1)–(46.4) and (45.1)–(45.4).

behavior of the methods is confirmed by the Log-log plots from Figures 1 and 3. The simulations presented in this work demonstrate a good agreement with respect to the analytical solutions employed. The proposed two-level method (29.1)–(29.2) gives better results for the EFK equation than those discussed in [16]. Using the fourth-order accurate schemes, we are able to simulate the dynamics of two-space-dimensional fourth-order problems with sparser grids and fewer time steps as compared with the standard second-order schemes. It is seen that the solutions become more accurate with decreasing values of time step and increasing values of grid size verifying the stability of the proposed methods for (P1) and (P2). The application of the derived methods to the singular equations of types (P1) and (P2) is discussed by modifying the methods in such a way that the solution retains its order and accuracy everywhere in the vicinity of the singularity. The proposed two-level compact operator schemes for linear equation of the kind (P3) are shown to be unconditionally stable. A further refinement allows us to obtain $O(k^2 + h^4)$ and $O(k^2 + h^2)$ ADI methods for (P3). Such ADI methods require the solution of only tri-diagonal systems of equations parallel to coordinate axes at each time step, thus facilitating the computations. According to the obtained results, the current approach is remarkably a successful technique for solving two-dimensional fourth-order nonlinear parabolic PDEs. The extension of the present approach may be useful to solve higher-dimensional problems appearing in various applications of science and engineering.

Acknowledgment: The authors would like to thank the anonymous referees for their extensive comments on the revision of the manuscript which substantially improved the quality of the paper.

References

[1] C. Andrade and S. McKee, High accuracy A.D.I. methods for fourth order parabolic equations with variable coefficients, *J. Comput. Appl. Math.* **3** (1977), 11–14.

[2] D. G. Aronson and H. F. Weinberger, Multidimensional nonlinear diffusion arising in population genetics, *Adv. Math.* **30** (1978), 33–76.

[3] M. J. Boussinesq, Theorie des ondes et des remous qui se propagent le long d’un canal rectangular horizontal, *J. Math. Pures Appl.* **17** (1872), 55–108.

[4] S. D. Conte, Numerical solution of vibration problems in two space variables, *Pacific J. Math.* **7** (1957), 1535–1544.
[5] S. H. Crandall, An optimum implicit recurrence formula for the heat conduction equation, *Quart. Appl. Math.* **13** (1955), 318–320.
[6] P. Danumjaya and A. K. Pani, Orthogonal cubic spline collocation method for the extended Fisher–Kolmogorov equation, *J. Comput. Appl. Math.* **174** (2005), 101–117.
[7] P. Danumjaya and A. K. Pani, Numerical methods for the extended Fisher–Kolmogorov (EFK) equation, *Int. J. Numer. Anal. Model.* **3** (2006), 186–210.
[8] G. T. Dee and W. van Saarloos, Bistable systems with propagating fronts leading to pattern formation, *Phys. Rev. Lett.* **60** (1988), 2641–2644.
[9] G. Fairweather and A. R. Gourlay, Some stable difference approximations to a fourth-order parabolic partial differential equation, *Math. Comp.* **21** (1967), 1–11.
[10] G. Fairweather and A. R. Mitchell, A high accuracy alternating direction method for the wave equation, *IMA J. Appl. Math.* **1** (1965), 309–316.
[11] G. Fairweather and A. R. Mitchell, A new computational procedure for A.D.I. methods, *SIAM J. Numer. Anal.* **4** (1967), 163–170.
[12] V. G. Gupta and S. Gupta, Analytical solution of singular fourth order parabolic partial differential equations of variable coefficients by using homotopy perturbation transform method, *J. Appl. Math. Inform.* **31** (2013), 165–177.
[13] L. A. Hageman and D. M. Young, *Applied Iterative Methods*, Dover Publication, New York, 2004.
[14] A. Kalogirou, E. E. Keaveny and D. T. Papageorgiou, An in-depth numerical study of the two-dimensional Kuramoto–Sivashinsky equation, *Proc. A* **471** (2015), no. 2179, DOI 10.1098/rspa.2014.0932.
[15] C. T. Kelly, *Iterative Methods for Linear and Non-Linear Equations*, SIAM Publications, Philadelphia, 1995.
[16] N. Khiari and K. Omrani, Finite difference discretization of the extended Fisher–Kolmogorov equation in two dimensions, *Comput. Math. Appl.* **62** (2011), 4151–4160.
[17] Y. Kuramoto and T. Tsuzuki, Persistent propagation of concentration waves in dissipative media far from thermal equilibrium, *Progr. Theor. Phys.* **55** (1976), 356–369.
[18] Y. Liu, H. Li, Z. Fang, S. He and J. Wang, A coupling method of new EMFE and FE for fourth-order partial differential equation of parabolic type, *Adv. Math. Phys.* **2013** (2013), Article ID 787891.
[19] R. K. Mohanty, A two level implicit difference formula of $O(k^2 + h^4)$ for the numerical solution of one space dimensional unsteady quasi-linear biharmonic problem of first kind, *J. Comput. Methods Sci. Eng.* **3** (2003), 193–208.
[20] R. K. Mohanty, A new high accuracy finite difference discretization for the solution of 2D nonlinear biharmonic equations using coupled approach, *Numer. Methods Partial Differential Equations* **26** (2010), 931–944.
[21] R. K. Mohanty, W. Dai and F. Han, A new high accuracy method for two-dimensional biharmonic equation with nonlinear third derivative terms: Application to Navier–Stokes equations of motion, *Int. J. Comput. Math.* **92** (2015), 1574–1590.
[22] R. K. Mohanty and D. J. Evans, Block iterative methos for the numerical solution of two dimensional non-linear biharmonic equations, *Int. J. Comput. Math.* **69** (1998), 371–389.
[23] R. K. Mohanty, D. J. Evans and D. Kumar, High accuracy difference formulae for a fourth order quasi-linear parabolic initial boundary value problem of first kind, *Int. J. Comput. Math.* **80** (2003), 381–398.
[24] R. K. Mohanty and D. Kaur, A class of quasi-variable mesh methods based on off-step discretization for the numerical solution of fourth-order quasi-linear parabolic partial differential equations, *Adv. Difference Equ.* **2016** (2016), Paper No. 326.
[25] R. K. Mohanty and D. Kaur, High accuracy implicit variable mesh methods for numerical study of special types of fourth order non-linear parabolic equations, *Appl. Math. Comput.* **273** (2016), 678–696.
[26] R. K. Mohanty and D. Kaur, Numerov type variable mesh approximations for 1D unsteady quasi-linear biharmonic problem: Application to Kuramoto–Sivashinsky equation, *Numer Algor.* (2016), DOI 10.1007/s11075-016-0154-3.
[27] R. K. Mohanty and P. K. Pandey, Difference methods of order two and four for systems of mildly nonlinear biharmonic problems of the second kind in two space dimensions, *Numer. Methods Partial Differential Equations* **12** (1996), 707–717.
[28] Y. Saad, *Iterative Methods for Sparse Linear Systems*, SIAM, Philadelphia, 2003.
[29] S. P. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells*, 2nd ed., McGraw–Hill, New York, 1959.
[30] E. H. Twizell and A. Q. M. Khaliq, A difference scheme with high accuracy in time for fourth-order parabolic equations, *Comput. Methods Appl. Mech. Engrg.* **41** (1983), 91–104.
[31] A.-M. Wazwaz, Exact solutions for variable coefficients fourth-order parabolic partial differential equations in higher-dimensional spaces, *Appl. Math. Comput.* **130** (2002), 415–424.
[32] G. B. Whitham, *Linear and Nonlinear Waves*, Wiley-Interscience, New York, 1974.
[33] T. P. Witelski and M. Bowen, ADI schemes for higher-order nonlinear diffusion equations, *Appl. Numer. Math.* **45** (2003), 331–351.
[34] G. Zhu, Experiments on director waves in nematic liquid crystals, *Phys. Rev. Lett.* **49** (1982), 1332–1335.