Applied Mathematical Modelling 55 (2018) 281–298

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Unconditionally stable high accuracy compact difference schemes for multi-space dimensional vibration problems with simply supported boundary conditions

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A R T I C L E   I N F O

Article history:
Received 18 December 2016
Revised 20 October 2017
Accepted 26 October 2017
Available online 11 November 2017

Keywords:
Euler–Bernoulli beam equation
Compact difference scheme
Two-level
Tridiagonal system
Unconditionally stable

A B S T R A C T

The Euler–Bernoulli beam equation is a fourth order parabolic partial differential equation governing the transverse vibrations of a long and slender beam and is thus of interest in various engineering applications. In this study, we propose new two-level implicit difference formulas for the solution of vibration problem in one, two and three space dimensions subject to appropriate initial and boundary conditions. The proposed methods are fourth order accurate in space and second order accurate in time and are based upon a single compact stencil. The boundary conditions are incorporated in a natural way without any discretization or introduction of fictitious nodes. The derived methods are shown to be unconditionally stable for model linear problems. Some physical examples and their numerical results are given to illustrate the accuracy of the proposed methods. The test problems confirm that the computed solutions are not only in good agreement with the exact solutions but also competent with the solutions derived in earlier research studies.

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1. Introduction

The beam is one of the primary components of an engineering structure. Vibration problems related to beams have various applications in building construction, in mechanical and aircraft designing and in earth sciences. The classical linear theory of deformation yields the Euler–Bernoulli beam equation which is a standard model that describes the dynamics of a long, thin and inextensible beam [1,2]. The solution of this equation represents transverse vibrations of the beam from an initial horizontal position. There exists a linear relationship between stress and strain for small transverse vibrations. Let us consider a beam of length L with its axis along the x-direction and $\phi(x, t)$ denotes the deflection of the beam from its axis. Then the governing equation of the one dimensional dynamic Euler–Bernoulli model is given by

$$\rho A(x)\frac{\partial^2\phi}{\partial t^2}+\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2\phi}{\partial x^2}\right)=f(x,t),\quad L_0<x<L_1,\ \ t>0, \tag{1}$$

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https://doi.org/10.1016/j.apm.2017.10.037
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subject to the initial conditions

$$\phi(x,0)=\phi_0(x),\quad \frac{\partial\phi}{\partial t}(x,0)=\phi_1(x),\quad L_0\le x\le L_1. \tag{2}$$

We assume that the beam is simply supported (or hinged) at its ends, consequently the displacement and bending moment are zero at the ends. For well posedness, it is not essential to impose homogenous boundary conditions and therefore we consider a generalization of the simply supported boundary conditions given by

$$\phi(L_0,t)=g_0(t),\quad \frac{\partial^2\phi}{\partial x^2}(L_0,t)=g_1(t),\quad t>0, \tag{3.1}$$

$$\phi(L_1,t)=h_0(t),\quad \frac{\partial^2\phi}{\partial x^2}(L_1,t)=h_1(t),\quad t>0, \tag{3.2}$$

where $L\equiv L_1-L_0$, ρ is the beam material density, $A(x)$ is the beam cross-section area, $EI(x)>0$ is referred as the beam bending stiffness or flexural rigidity and $f(x, t)$ is the load per unit length (source term). The initial and boundary conditions are given with sufficient smoothness in order to maintain the order and accuracy of the difference schemes under consideration. It is assumed that shear deformations and effects of rotary inertia are negligible so that the equation of motion is obtained from Euler–Bernoulli model rather than the Timoshenko model (thick beam theory).

The solution of this kind of variable coefficient fourth order parabolic partial differential equation (PDE) is of interest in several engineering applications ranging from the construction of supporting structures or axles in railway construction to the development of robotic designs. As the dynamic beam equation (1) involves a fourth order spatial derivative, the boundary treatment is non-trivial and hence it is required to develop methods that can impose physical boundary conditions and still guarantee stability. Both explicit and implicit methods have been proposed for the solution of beam equation [3–6]. The explicit method of Collatz [3] is based on 7-point scheme but suffers from severe stability restriction. The scheme suggested by Crandall [4] is implicit but is based on 11 points while the implicit scheme given by Conte [6] is based on 15 points. The formula given by Albrecht [5] overcomes stability restriction but is computationally disadvantageous. Numerical solution of equation (1) by decomposing it into a system of second-order equations have been studied by Evans [7], Fairweather and Gourlay [8] and Jain et al. [9]. Some discrete Galerkin methods were formulated for approximating a two dimensional linear fourth order parabolic PDE in [10]. Andrade and McKee [11] derived high accuracy alternating direction implicit methods for fourth order parabolic equations with variable coefficients in one, two and three space dimensions based on a non-compact stencil. Twizell and Khaliq [12] evolved a finite difference method from a multiderivative method for second-order ordinary differential equations for the numerical solution of fourth order parabolic PDEs in one and two space variables. Saul'ev [13] constructed an explicit second order convergent three-layer approximation for fourth order m-dimensional parabolic equation with a weak limitation on stability. Khaliq [14] developed a predictor–corrector method for the numerical solution of a homogenous beam hinged at both ends. A fully Sinc–Galerkin method in both space and time for fourth order time-dependent PDEs with fixed and cantilever boundary conditions was proposed by Smith et al. [15]. High order implicit difference formulas using a combination of values of $\phi(x, t)$ and $\phi_x(x, t)$ for the clamped beam vibration were reported in [16]. Later, Aziz et al. [17] and Khan et al. [18] developed three level schemes based on spline collocation. Caglar and Caglar [19] have proposed a fifth degree B-spline method to solve beam equation (1). Rashidinia and Mohammadi [20] reported spatially fourth order accurate three-level implicit methods using sextic spline in space and finite difference in time directions. Mittal and Jain [21] presented schemes based on cubic and quintic B-spline functions to create two and three-level implicit methods for solving constant coefficient fourth order parabolic equations. Recently, Mohammadi [22] proposed an implicit three level second-order convergent in space direction a sextic B-spline difference scheme.

Among various numerical methods available to approximate a fourth order parabolic PDE such as the finite element method, spline collocation method, the finite difference method is attractive because of its relative ease of implementation, flexibility and accuracy in the solution values. The finite difference method broke the area or volume of interest into a grid system on which the partial derivatives can be expressed as finite differences. The problem then reduces to finding the solution at the grid points as a set of linear algebraic equations called the finite difference equations. However, higher order approximation on a non-compact cell increases the bandwidth of the coefficient matrix resulting from the discretization method and ultimately increased bandwidth leads to more arithmetic operations. Thus neither a lower order method on a fine mesh nor a higher order method on a non-compact cell is computationally efficient. Hence, higher order compact difference methods which are restricted to the patch of cells immediately surrounding any given grid point are gradually acquiring recognition. Apart from this, the advantage of developing a compact scheme is its suitability to be used directly adjacent to the boundary without introducing any extra nodes outside the boundary of the domain. High order compact finite difference schemes have been developed for solving several important PDEs having physical relevance, see e.g., [23–25] where supporting numerical studies showing the higher-order rates of convergence were presented. In general, a direct discretization of fourth order partial derivative involves at least five grid points. Recently, Mohanty and Kaur [26–29] proposed three-level and two-level compact implicit finite difference methods for two different classes of nonlinear fourth order parabolic PDEs in one and two space variables using three and nine spatial grid points, respectively. In this paper, we present two-level implicit difference formulas of order two in time and four in space using 3-point, 9-point and 19-point compact cell for the solution of vibration problem in one, two and three space variables, respectively. The given boundary conditions are exactly satisfied and no approximations for derivatives need to be carried out at the boundaries in the present approach.

The layout of the paper is as follows. In Section 2, we derive the unconditionally stable method for Euler–Bernoulli beam equation using three spatial grid points at each time level. In Section 3, we discuss the scheme for two dimensional vibration problem and it is shown to be unconditionally stable. We extend our approach for three dimensional problem in Section 4 and present the stability analysis. In Section 5, we solve multidimensional vibration problems using the proposed methods to confirm the correctness of theoretical considerations and wherever possible, the results are compared with the other existing methods. Conclusions are given in Section 6.

2. Numerical formulation for one dimensional vibration problem and stability analysis

Following Fairweather and Gourlay [8], we introduce two new variables defined by

$$u = \frac{\partial \phi}{\partial t}, \quad v = EI(x)\frac{\partial^2 \phi}{\partial x^2}$$

Then the beam equation (1) can be rewritten as two simultaneous PDEs of second-order in the form:

$$\frac{\partial^2 v}{\partial x^2} = -\rho A(x)\frac{\partial u}{\partial t} + f(x,t), \quad (x,t) \in \Omega_1 \tag{4.1}$$

$$EI(x)\frac{\partial^2 u}{\partial x^2} = \frac{\partial v}{\partial t}, \quad (x,t) \in \Omega_1 \tag{4.2}$$

where $\Omega_1 = \{(x,t) : L_0 < x < L_1,\ t > 0\}$. It is noted that since the grid lines are parallel to the coordinate axes and ϕ is exactly known on the boundary of the solution region Ω_1, this implies that all successive tangential partial derivatives of ϕ are known there. At $t = 0$, ϕ is prescribed so its tangential partial derivatives $\phi_x, \phi_{xx}, \ldots$ are known at $t = 0$, i.e., the values of u and v are known at $t = 0$. Likewise, u and v are known on all sides of Ω_1.

In order to determine the numerical solution of the above initial boundary value problem, we divide the interval $[L_0, L_1]$ into $(N+1)$ subintervals each of length $h > 0$ so that $L_1 - L_0 = (N+1)h$, N being a positive integer. Let $\tau > 0$ be the step size in the temporal direction and $\lambda = (\tau/h^2) > 0$ be the mesh ratio parameter. The solution region Ω_1 is replaced by a set of grid points $(x_l, t_j) = (L_0 + lh, j\tau)$, $l = 0, \ldots, N+1$, $j = 0, 1, \ldots$. Let U_l^j, V_l^j be the exact solution values and u_l^j, v_l^j be the approximate values of $u(x, t)$, $v(x,t)$, respectively at the nodal point (x_l, t_j). For $S = U, V$, we denote

$$S_{ab} = \left(\frac{\partial^{a+b} S}{\partial x^a \partial t^b}\right)_{(x_l,t_j)}, \quad a, b = 0, 1, 2, \ldots$$

Consider the following approximations:

$$\widehat{t}_j = t_j + \theta\tau, \quad 0 < \theta < 1, \tag{5.1}$$

$$\widehat{U}_l^j = \theta U_l^{j+1} + (1-\theta)U_l^j = U_l^j + \theta\tau U_{01} + O(\tau^2), \tag{5.2}$$

$$\widehat{U}_{l\pm1}^j = \theta U_{l\pm1}^{j+1} + (1-\theta)U_{l\pm1}^j = U_{l\pm1}^j + \theta\tau U_{01} \pm \theta\tau h U_{11} + O(\tau h^2 + \tau^2), \tag{5.3}$$

$$\widehat{U}_{t_l}^j = (U_l^{j+1} - U_l^j)/\tau = U_{t_l}^j + \frac{\tau}{2}U_{02} + O(\tau^2), \tag{5.4}$$

$$\widehat{U}_{t_{l\pm1}}^j = (U_{l\pm1}^{j+1} - U_{l\pm1}^j)/\tau = U_{t_{l\pm1}}^j + \frac{\tau}{2}U_{02} \pm \frac{\tau h}{2}U_{12} + O(\tau h^2 + \tau^2), \tag{5.5}$$

$$\widehat{U}_{xx_l}^j = (\widehat{U}_{l+1}^j - 2\widehat{U}_l^j + \widehat{U}_{l-1}^j)/h^2 = U_{xx_l}^j + \theta\tau U_{21} + O(\tau^2 + h^2), \tag{5.6}$$

$$\widehat{F}_l^j = f(x_l, \widehat{t}_j) = F_l^j + \theta\tau\alpha_l^j + O(\tau^2), \tag{5.7}$$

$$\widehat{F}_{l\pm1}^j = f(x_{l\pm1}, \widehat{t}_j) = F_{l\pm1}^j + \theta\tau\alpha_l^j \pm \theta\tau h\beta_l^j + O(\tau h^2 + \tau^2), \tag{5.8}$$

where $F_{l\pm r}^j = f(x_{l\pm r}, t_j)$, $r = 0, \pm1$, $\alpha_l^j = \left(\frac{\partial f}{\partial t}\right)_{(x_l,t_j)}$ and $\beta_l^j = \left(\frac{\partial^2 f}{\partial x \partial t}\right)_{(x_l,t_j)}$. Similar approximations are defined for the solution variable $v(x,t)$.

A difference method of $O(\tau^2+h^2)$ for the system (4.1)–(4.2) may be written as

$$\widehat{V}^j_{xx_l} = -\rho A_l \widehat{U}^j_{t_l} + \widehat{F}^j_l + O(\tau^2+h^2), \tag{6.1}$$

$$EI_l\,\widehat{U}^j_{xx_l} = \widehat{V}^j_{t_l} + O(\tau^2+h^2),\ l=1,\dots,N,\ j=0,1,\dots \tag{6.2}$$

for $\theta=1/2$, where $A_l = A(x_l)$ and $EI_l = EI(x_l)$. The displacement $\phi(x, t)$ at the point (x_l, t_j) may be obtained from the values of U^j_l and V^j_l by means of the algorithm

$$EI_l\left(\phi^{j+1}_{l+1} - 2\phi^{j+1}_l + \phi^{j+1}_{l-1}\right) = h^2 V^{j+1}_l + O(h^4),\ l=1,\dots,N,\ j=0,1,\dots \tag{7}$$

A difference method of $O(\tau^2+\tau h^2+h^4)$ for the system (4.1)–(4.2) may be written as

$$\delta_x^2\widehat{V}^j_l = -\frac{\rho h^2}{12}\left[A_l(12+\delta_x^2) + hA_{x_l}(2\mu_x\delta_x) + h^2 A_{xx_l}\right]\widehat{U}^j_{t_l} + \frac{h^2}{12}\left[\widehat{F}^j_{l+1} + 10\widehat{F}^j_l + \widehat{F}^j_{l-1}\right] + \widehat{T}^{j^{(1)}}_l, \tag{8.1}$$

$$\begin{aligned}&\left[6EI_l - h^2\left(\frac{EI_{x_l}}{EI_l}\right)EI_{x_l} + \frac{h^2}{2}EI_{xx_l}\right]\delta_x^2\widehat{U}^j_l\\ &= \frac{h^2}{2}\left[\left(1-\frac{hEI_{x_l}}{EI_l}\right)\widehat{V}^j_{t_{l+1}} + 10\widehat{V}^j_{t_l} + \left(1+\frac{hEI_{x_l}}{EI_l}\right)\widehat{V}^j_{t_{l-1}}\right] + \widehat{T}^{j^{(2)}}_l,\ l=1,\dots,N,\ j=0,1,\dots\end{aligned} \tag{8.2}$$

where $\theta=1/2$ and the local truncation errors are given by $\widehat{T}^{j^{(1)}}_l = O(\tau^2h^2+\tau h^4+h^6)$ and $\widehat{T}^{j^{(2)}}_l = O(\tau^2h^2+\tau h^4+h^6)$. Here δ_x and μ_x are the central and average difference operators, respectively with respect to x-direction. When $\tau\propto h^2$, that is, for a fixed value of λ, the method (8.1)–(8.2) become $O(h^4)$ accurate in space.

Differentiating (4.1)–(4.2) with respect to 't' at the grid point (x_l, t_j), we obtain

$$V_{21} = -\rho A_l U_{02} + \alpha^j_l, \tag{9.1}$$

$$EI_l\,U_{21} = V_{02} \tag{9.2}$$

By the help of approximations (5.1)–(5.8), from (8.1), we obtain

$$\begin{aligned}\delta_x^2 V^j_l + \theta\tau h^2 V_{21} + O(\tau^2h^2+\tau h^4) &= -\frac{\rho h^2}{12}\left[A_l(12+\delta_x^2) + hA_{x_l}(2\mu_x\delta_x) + h^2A_{xx_l}\right]U^j_{t_l}\\ &\quad + \frac{h^2}{12}\left[F^j_{l+1} + 10F^j_l + F^j_{l-1}\right] - \frac{\rho\tau h^2}{2}A_lU_{02} + \theta\tau h^2\alpha^j_l + \widehat{T}^{j^{(1)}}_l\end{aligned} \tag{10}$$

Further the following relation holds:

$$\delta_x^2 V^j_l = -\frac{\rho h^2}{12}\left[A_l(12+\delta_x^2) + hA_{x_l}(2\mu_x\delta_x) + h^2A_{xx_l}\right]U^j_{t_l} + \frac{h^2}{12}\left[F^j_{l+1} + 10F^j_l + F^j_{l-1}\right] + O(h^6) \tag{11}$$

Substituting (9.1) into (10) and using the relation (11), we obtain

$$\widehat{T}^{j^{(1)}}_l = \left(\theta - \frac{1}{2}\right)\rho\tau h^2 A_l U_{02} + O(\tau^2h^2+\tau h^4+h^6) \tag{12}$$

For the proposed method (8.1) to be $O(\tau^2+\tau h^2+h^4)$, the coefficient of τh^2 in (12) must be zero. Thus we obtain $\theta = 1/2$ and the local truncation error reduces to $\widehat{T}^{j^{(1)}}_l = O(\tau^2h^2+\tau h^4+h^6)$.

Using the approximations (5.1)–(5.6), the relation (9.2) and $\theta=1/2$, the local truncation error reduces $\widehat{T}^{j^{(2)}}_l$ associated with (8.2) is obtained as $\widehat{T}^{j^{(2)}}_l = O(\tau^2h^2+\tau h^4+h^6)$.

Finally, the displacement $\phi(x, t)$ at the grid point (x_l, t_j) may be obtained from the values of U^j_l and V^j_l by means of the following Numerov type algorithm:

$$\begin{aligned}&\left[6EI_l - h^2\left(\frac{EI_{x_l}}{EI_l}\right)EI_{x_l} + \frac{h^2}{2}EI_{xx_l}\right]\left(\phi^{j+1}_{l+1} - 2\phi^{j+1}_l + \phi^{j+1}_{l-1}\right)\\ &= \frac{h^2}{2}\left[\left(1-\frac{hEI_{x_l}}{EI_l}\right)V^{j+1}_{l+1} + 10V^{j+1}_l + \left(1+\frac{hEI_{x_l}}{EI_l}\right)V^{j+1}_{l-1}\right] + O(h^6),\ l=1,\dots,N,\ j=0,1,\dots\end{aligned} \tag{13}$$

For a homogenous beam, both EI and A are constants, then Eq. (1) reduces to

$$\frac{\partial^2\phi}{\partial t^2} + c^2\frac{\partial^4\phi}{\partial x^4} = \frac{f(x,t)}{\rho A}, \quad (x,t) \in \Omega_1, \tag{14}$$

where $c = \sqrt{\frac{EI}{\rho A}}$.

Now, we consider a fourth order parabolic equation with variable coefficient of the form considered in [11]:

$$\frac{\partial^2\phi}{\partial t^2} + B(x,t)\frac{\partial^4\phi}{\partial x^4} = f(x,t), \quad (x,t) \in \Omega_1. \tag{15}$$

For this case, u and v are defined as

$$u = \frac{\partial\phi}{\partial t}, \quad v = \frac{\partial^2\phi}{\partial x^2}$$

The variable coefficient fourth order parabolic PDE (15) can be rewritten as two simultaneous PDEs of second-order in the form:

$$B(x,t)\frac{\partial^2 v}{\partial x^2} = -\frac{\partial u}{\partial t} + f(x,t), \tag{16.1}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial v}{\partial t} \tag{16.2}$$

A central difference method of $O(\tau^2+h^2)$ for the system (16.1)–(16.2) may be written as

$$\widehat{B}_l^j\widehat{V}_{xx_l}^j = -\widehat{U}_{t_l}^j + \widehat{F}_l^j + O(\tau^2+h^2), \tag{17.1}$$

$$\widehat{U}_{xx_l}^j = \widehat{V}_{t_l}^j + O(\tau^2+h^2), \tag{17.2}$$

for $\theta = 1/2$ and $\widehat{B}_l^j = B(x_l, \widehat{t}_j)$. This method in conjunction with the following second order scheme is used to calculate the value of $\phi(x, t)$ at the grid point (x_l, t_j):

$$\phi_{l+1}^{j+1} - 2\phi_l^{j+1} + \phi_{l-1}^{j+1} = h^2 V_l^{j+1} + O(h^4), \quad l = 1,\dots,N, \ \ j = 0,1,\dots \tag{18}$$

A difference method of $O(\tau^2 + \tau h^2 + h^4)$ for the system (16.1)–(16.2) may be written as

$$\begin{aligned}\left[6\widehat{B}_l^j - h^2\left(\frac{\widehat{B}_{x_l}^j}{\widehat{B}_l^j}\right)\widehat{B}_{x_l}^j + \frac{h^2}{2}\widehat{B}_{xx_l}^j\right]\delta_x^2\widehat{V}_l^j = &-\frac{h^2}{2}\left[\left(1-\frac{h\widehat{B}_{x_l}^j}{\widehat{B}_l^j}\right)\widehat{U}_{t_{l+1}}^j + 10\widehat{U}_{t_l}^j + \left(1+\frac{h\widehat{B}_{x_l}^j}{\widehat{B}_l^j}\right)\widehat{U}_{t_{l-1}}^j\right] \\ &+\frac{h^2}{2}\left[\left(1-\frac{h\widehat{B}_{x_l}^j}{\widehat{B}_l^j}\right)\widehat{F}_{l+1}^j + 10\widehat{F}_l^j + \left(1+\frac{h\widehat{B}_{x_l}^j}{\widehat{B}_l^j}\right)\widehat{F}_{l-1}^j\right] \\ &+O(\tau^2h^2 + \tau h^4 + h^6),\end{aligned} \tag{19.1}$$

$$\delta_x^2\widehat{U}_l^j = \frac{h^2}{12}\left[\widehat{V}_{t_{l+1}}^j + 10\widehat{V}_{t_l}^j + \widehat{V}_{t_{l-1}}^j\right] + O(\tau^2h^2 + \tau h^4 + h^6), \tag{19.2}$$

for $\theta = 1/2$ and $\widehat{B}_{x_l}^j = B_x(x_l, \widehat{t}_j)$ etc.

Using the technique discussed in [28], we obtain the following fourth order difference scheme to (16.1):

$$\begin{aligned}&\left[6B_l^j - h^2\left(\frac{B_{x_l}^j}{B_l^j}\right)B_{x_l}^j + \frac{h^2}{2}B_{xx_l}^j\right]\delta_x^2 V_l^j \\ &= -\frac{h^2}{2}\left[\left(1-\frac{hB_{x_l}^j}{B_l^j}\right)U_{t_{l+1}}^j + 10U_{t_l}^j + \left(1+\frac{hB_{x_l}^j}{B_l^j}\right)U_{t_{l-1}}^j\right] \\ &+\frac{h^2}{2}\left[\left(1-\frac{hB_{x_l}^j}{B_l^j}\right)F_{l+1}^j + 10F_l^j + \left(1+\frac{hB_{x_l}^j}{B_l^j}\right)F_{l-1}^j\right] + O(h^6).\end{aligned} \tag{20}$$

Using the approximations (5.1)–(5.8) in (19.1) and by the help of Taylor's series, we obtain

$$\left[6B_l^j - h^2\left(\frac{B_{x_l}^j}{B_l^j}\right)B_{x_l}^j + \frac{h^2}{2}B_{xx_l}^j\right]\delta_x^2 V_l^j + 6\theta\tau h^2\left(B_l^j V_{21} + B_{t_l}^j V_{20}\right) + O(\tau^2h^2 + \tau h^4)$$
$$= -\frac{h^2}{2}\left[\left(1 - \frac{hB_{x_l}^j}{B_l^j}\right)U_{t_{l+1}}^j + 10U_{t_l}^j + \left(1 + \frac{hB_{x_l}^j}{B_l^j}\right)U_{t_{l-1}}^j\right] - 3\tau h^2 U_{02}$$
$$+ \frac{h^2}{2}\left[\left(1 - \frac{hB_{x_l}^j}{B_l^j}\right)F_{l+1}^j + 10F_l^j + \left(1 + \frac{hB_{x_l}^j}{B_l^j}\right)F_{l-1}^j\right] + 6\theta\tau h^2\alpha_l^j + (\mathrm{LTE})_1 \tag{21}$$

where $(\mathrm{LTE})_1$ is the local truncation error associated with (19.1). Differentiating (16.1) with respect to 't' at the grid point (x_l, t_j), we obtain

$$B_l^j V_{21} + B_{t_l}^j V_{20} = -U_{02} + \alpha_l^j \tag{22}$$

Using the above relation and the difference formula (20) in (21), the local truncation error $(\mathrm{LTE})_1$ associated with (19.1) is obtained as

$$(\mathrm{LTE})_1 = 6\left(\theta - \frac{1}{2}\right)\tau h^2 U_{02} + O(\tau^2h^2 + \tau h^4 + h^6) \tag{23}$$

The proposed method (19.1) is of $O(\tau^2 + \tau h^2 + h^4)$ if the coefficient of τh^2 in (23) is zero. Thus we obtain $\theta = 1/2$ and the local truncation error reduces to $(\mathrm{LTE})_1 = O(\tau^2h^2 + \tau h^4 + h^6)$. Similarly, the local truncation error $(\mathrm{LTE})_2$ associated with (19.2) is obtained as $(\mathrm{LTE})_2 = O(\tau^2h^2 + \tau h^4 + h^6)$.

Using the following Numerov type scheme, the displacement $\phi(x, t)$ at the grid point (x_l, t_j) is obtained as

$$\phi_{l+1}^{j+1} - 2\phi_l^{j+1} + \phi_{l-1}^{j+1} = \frac{h^2}{12}\left[V_{l+1}^{j+1} + 10V_l^{j+1} + V_{l-1}^{j+1}\right] + O(h^6), \quad l = 1, \ldots, N, \quad j = 0, 1, \ldots \tag{24}$$

To discuss the stability, we consider the following linear fourth order parabolic PDE in one space variable:

$$\frac{\partial^2\phi}{\partial t^2} + \frac{\partial^4\phi}{\partial x^4} = f(x,t), \quad L_0 < x < L_1, \quad t > 0, \tag{25}$$

which is the governing equation of forced transverse vibration of a uniform beam.

Applying methods (17.1)–(17.2) and (19.1)–(19.2) to the above equation results in the following block tridiagonal matrix structure:

$$\boldsymbol{A}\boldsymbol{y}^{j+1} = \boldsymbol{B}\boldsymbol{y}^j + \boldsymbol{c}, \quad j = 0, 1, \ldots \tag{26}$$

where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ -\boldsymbol{A}_2 & \boldsymbol{A}_1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{A}_1 & -\boldsymbol{A}_2 \\ \boldsymbol{A}_2 & \boldsymbol{A}_1 \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix}, \quad \boldsymbol{c} = \begin{bmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \end{bmatrix}$$

The matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are $2N \times 2N$ block tridiagonal while $\boldsymbol{y}$ and $\boldsymbol{c}$ are column vectors of order $2N$. Also, $\boldsymbol{u} = (u_1, u_2, \ldots, u_N)^T$, $\boldsymbol{v} = (v_1, v_2, \ldots, v_N)^T$ are solution vectors; $\boldsymbol{c_1}$, $\boldsymbol{c_2}$ are right-hand side column vectors of order N that consist of homogenous functions, initial and boundary values of the block system (26). The submatrices for the method (17.1)–(17.2) are given by

$$\boldsymbol{A}_1 = [0, 1, 0], \ \boldsymbol{A}_2 = \frac{\lambda}{2}[1, -2, 1]$$

and the submatrices for the method (19.1)–(19.2) are given by

$$\boldsymbol{A}_1 = [1, 10, 1], \ \boldsymbol{A}_2 = 6\lambda[1, -2, 1],$$

where $[a, b, c]$ denotes the $N \times N$ tridiagonal matrix whose eigenvalues are given by $b + 2\sqrt{ac}\cos(2\sigma_l)$, $2\sigma_l = (l\pi)/(N+1)$, $l = 1, \ldots, N$. For discussing the stability, we consider the homogenous part of (26), which may be written as

$$\boldsymbol{y}^{j+1} = (\boldsymbol{A}^{-1}\boldsymbol{B})\boldsymbol{y}^j, \quad j = 0, 1, \ldots \tag{27}$$

Let $\boldsymbol{\varepsilon}^j = \boldsymbol{y}^j - \boldsymbol{Y}^j$ be the error vector at the jth iterate (in the absence of round-off errors), where $\boldsymbol{Y}^j = \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \end{bmatrix}^j$. The error equation may be written as

$$\boldsymbol{\varepsilon}^{j+1} = \boldsymbol{H}\boldsymbol{\varepsilon}^j, \quad j = 0, 1, \ldots$$

where $\boldsymbol{H} = \boldsymbol{A}^{-1}\boldsymbol{B}$ is the amplification matrix.

The eigenvalues of $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ for method (17.1)–(17.2) are given by $\alpha_l = 1$ and $\beta_l = -2\lambda\sin^2\sigma_l$ for $l = 1, \ldots, N$, respectively. The submatrix $\boldsymbol{A_2}$ has N different eigenvalues so it has N linearly independent eigenvectors v_l, $l = 1, \ldots, N$. Although the submatrix $\boldsymbol{A_1}$ has N eigenvalues each equal to 1, it has N linearly independent eigenvectors which may be taken as v_l, $l=1,\ldots,N$ since the eigenvalue equation $\boldsymbol{A_1}v_l = 1.v_l$ is satisfied. Hence the eigenvalues of matrix $\boldsymbol{A}$ are the eigenvalues of the matrix

$$\begin{bmatrix} \alpha_l & \beta_l \\ -\beta_l & \alpha_l \end{bmatrix}$$

where α_l and β_l are the lth eigenvalues of $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$, respectively (see [30]). Thus, the eigenvalues of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ for the difference method (17.1)–(17.2) are given by $\alpha_l \pm \mathrm{i}\beta_l$, $l=1,\ldots,N$.

The eigenvalues of $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ for method (19.1)–(19.2) are given by $\eta_l = 12 - 4\sin^2\sigma_l$ and $\gamma_l = -24\lambda\sin^2\sigma_l$ for $l = 1,\ldots,N$, respectively. In this case, the submatrices $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ commute and have linear elementary divisors so they have a common system of eigenvectors. The eigenvalues of $\boldsymbol{A}$ are the eigenvalues of the matrices

$$\begin{bmatrix} \eta_l & \gamma_l \\ -\gamma_l & \eta_l \end{bmatrix}$$

Thus, the eigenvalues of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ for the difference method (19.1)–(19.2) are given by $\eta_l \pm \mathrm{i}\gamma_l$, $l=1,\ldots,N$ (see [30]).

Since the matrices $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}$ commute each other, eigenvalues of the amplification matrix $\boldsymbol{H}$ are given by:

$$\xi_{1_l} = \frac{\alpha_l + \mathrm{i}\beta_l}{\alpha_l - \mathrm{i}\beta_l},\quad \xi_{2_l} = \frac{\alpha_l - \mathrm{i}\beta_l}{\alpha_l + \mathrm{i}\beta_l},\quad l=1,\ldots,N,$$

for method (17.1)–(17.2) and for the method (19.1)–(19.2):

$$\xi_{1_l} = \frac{\eta_l + \mathrm{i}\gamma_l}{\eta_l - \mathrm{i}\gamma_l},\quad \xi_{2_l} = \frac{\eta_l - \mathrm{i}\gamma_l}{\eta_l + \mathrm{i}\gamma_l},\quad l=1,\ldots,N.$$

It is noted that $|\xi_{i_l}| = 1$, $l=1,\ldots,N$ for $i=1,2$.

For the matrix method of stability analysis, the Lax–Richtmyer condition [30], $\|\boldsymbol{H}^j\| \le M$, $j=1,2,\ldots,$ where M is a positive number independent of h, τ and j is equivalent to $\max\limits_{l=1,\ldots,N} |\xi_{i_l}| \le 1$, $i=1,2$. Since the magnitude of all the eigenvalues of $\boldsymbol{H}$ is equal to one, there is no condition on h and τ that will necessarily cause an instability. Further, it is noted that since the matrices $\boldsymbol{A^{-1}}$ and $\boldsymbol{B}$ commute each other, it follows that $\boldsymbol{H}$ is an orthogonal matrix, so $\|\boldsymbol{H}\|_2 = \sqrt{\rho(\boldsymbol{H}^T\boldsymbol{H})} = \sqrt{\rho(\boldsymbol{H}^{-1}\boldsymbol{H})} = \sqrt{\rho(\boldsymbol{I})} = 1$. The Lax–Richtmyer definition of stability is satisfied by the amplification matrix $\boldsymbol{H}$. Thus the proposed methods (17.1)–(17.2) and (19.1)–(19.2) are unconditionally stable.

We see that the proposed difference methods (17.1)–(17.2) and (19.1)–(19.2) are consistent with the differential equations (16.1)–(16.2). Hence by *Lax Equivalence Theorem* [30], the methods (17.1)–(17.2) and (19.1)–(19.2) are convergent for uniform beam vibration (25).

3. Difference scheme and stability for two dimensional vibration problem

We consider the problem of determining transverse vibrations of a thin plate of variable density or cross section hinged at its boundaries. The governing equation is

$$\frac{\partial^2\phi}{\partial t^2} + A(x,y,t)\left(\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{\partial^4\phi}{\partial y^4}\right) = f(x,y,t),\quad L_0 < x,y < L_1,\quad t>0, \tag{28}$$

The initial conditions associated with (28) are

$$\phi(x,y,0) = \phi_{00}(x,y),\quad \phi_t(x,y,0) = \phi_{11}(x,y),\quad L_0 \le x,y \le L_1 \tag{29}$$

and the boundary conditions are given by

$$\phi(x,L_0,t) = a_{10}(x,t),\quad \phi_{yy}(x,L_0,t) = a_{20}(x,t),\quad L_0 \le x \le L_1,\quad t>0 \tag{30.1}$$

$$\phi(x,L_1,t) = a_{11}(x,t),\quad \phi_{yy}(x,L_1,t) = a_{21}(x,t),\quad L_0 \le x \le L_1,\quad t>0 \tag{30.2}$$

$$\phi(L_0,y,t) = b_{01}(y,t),\quad \phi_{xx}(L_0,y,t) = b_{02}(y,t),\quad L_0 \le y \le L_1,\quad t>0 \tag{30.3}$$

$$\phi(L_1,y,t) = b_{11}(y,t),\quad \phi_{xx}(L_1,y,t) = b_{12}(y,t),\quad L_0 \le y \le L_1,\quad t>0 \tag{30.4}$$

Let $u = \frac{\partial\phi}{\partial t}$, $v = \nabla^2\phi$, where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ represents the two space dimensional Laplacian. Then Eq. (28) may be written as second-order system:

$$A(x,y,t)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = -\frac{\partial u}{\partial t} + f(x,y,t), \tag{31.1}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial v}{\partial t} \tag{31.2}$$

Let $\Omega_2 = \{(x, y, t) : L_0 < x, y < L_1,\ t > 0\}$. Note that since the grid lines are parallel to coordinate axes and ϕ is exactly known at the boundary, this imples that all successive tangential partial derivatives of ϕ are known at the boundary of Ω_2. For instance, at $y = L_0$, the values of ϕ and ϕ_{yy} are given, then the values ϕ_x, $\phi_{xx}, \ldots$ are determined at $y = L_0$. This implies that $\nabla^2\phi \equiv \phi_{xx} + \phi_{yy}$ is known at $y = L_0$. In a similar way, the values of u and v are known on all sides of Ω_2.

The solution region Ω_2 is covered by a rectilinear mesh with spacing $h > 0$ and $\tau > 0$ in the space and time coordinates, respectively. The grid points are given by (x_l, y_m, t_j) where $x_l = L_0 + lh,\ y_m = L_0 + mh;\ l, m = 0, 1, \ldots, N+1$ with $(N+1)h = L_1 - L_0$ and $t_j = j\tau,\ j = 0, 1, \ldots$. Let $U^j_{l,m},\ V^j_{l,m}$ be the exact solution values and $u^j_{l,m},\ v^j_{l,m}$ be the approximate values of $u(x, y, t)$, $v(x, y, t)$, respectively at the nodal point (x_l, y_m, t_j).

Consider the following approximations, for $p, q = 0, \pm 1$:

$$\widehat{U}^j_{l+p,m+q} = \left(U^{j+1}_{l+p,m+q} + U^j_{l+p,m+q}\right)/2, \tag{32.1}$$

$$\widehat{U}^j_{t_{l+p,m+q}} = \left(U^{j+1}_{l+p,m+q} - U^j_{l+p,m+q}\right)/\tau, \tag{32.2}$$

$$\widehat{U}^j_{xx_{l,m}} = \left(\widehat{U}^j_{l+1,m} - 2\widehat{U}^j_{l,m} + \widehat{U}^j_{l-1,m}\right)/h^2, \tag{32.3}$$

$$\widehat{U}^j_{yy_{l,m}} = \left(\widehat{U}^j_{l,m+1} - 2\widehat{U}^j_{l,m} + \widehat{U}^j_{l,m-1}\right)/h^2, \tag{32.4}$$

$$\widehat{F}^j_{l+p,m+q} = f(x_{l+p}, y_{m+q}, \widehat{t}_j). \tag{32.5}$$

An $O(\tau^2 + h^2)$ method for (31.1)–(31.2) may be written as

$$\widehat{A}^j_{l,m}\left(\widehat{V}^j_{xx_{l,m}} + \widehat{V}^j_{yy_{l,m}}\right) = -\widehat{U}^j_{t_{l,m}} + \widehat{F}^j_{l,m} + O(\tau^2 + h^2), \tag{33.1}$$

$$\widehat{U}^j_{xx_{l,m}} + \widehat{U}^j_{yy_{l,m}} = \widehat{V}^j_{t_{l,m}} + O(\tau^2 + h^2), \tag{33.2}$$

where $\widehat{A}^j_{l,m} = A(x_l, y_m, \widehat{t}_j)$. The above difference scheme in conjunction with the following method is used for obtaining $\phi^{j+1}_{l,m}$ at each time step:

$$\phi^{j+1}_{l+1,m} + \phi^{j+1}_{l-1,m} + \phi^{j+1}_{l,m+1} + \phi^{j+1}_{l,m-1} - 4\phi^{j+1}_{l,m} = h^2 V^{j+1}_{l,m} + O(h^4),\quad l, m = 1, \ldots, N,\ j = 0, 1, \ldots \tag{34}$$

An $O(\tau^2 + \tau h^2 + h^4)$ method for (31.1)–(31.2) may be written as

$$\begin{aligned}
&\left[\left(6\widehat{A}^j_{l,m} - h^2\left(\frac{\widehat{A}^j_{x_{l,m}}}{\widehat{A}^j_{l,m}}\right)\widehat{A}^j_{x_{l,m}} - h^2\left(\frac{\widehat{A}^j_{y_{l,m}}}{\widehat{A}^j_{l,m}}\right)\widehat{A}^j_{y_{l,m}} + \frac{h^2}{2}\left(\widehat{A}^j_{xx_{l,m}} + \widehat{A}^j_{yy_{l,m}}\right)\right)(\delta_x^2 + \delta_y^2) + \widehat{A}^j_{l,m}\delta_x^2\delta_y^2\right]\widehat{V}^j_{l,m} \\
&= \frac{h^2}{2}\left[\left(1 - \frac{h\widehat{A}^j_{x_{l,m}}}{\widehat{A}^j_{l,m}}\right)\left(\widehat{F}^j_{l+1,m} - \widehat{U}^j_{t_{l+1,m}}\right) + \left(1 + \frac{h\widehat{A}^j_{x_{l,m}}}{\widehat{A}^j_{l,m}}\right)\left(\widehat{F}^j_{l-1,m} - \widehat{U}^j_{t_{l-1,m}}\right)\right. \\
&\quad \left. + \left(1 - \frac{h\widehat{A}^j_{y_{l,m}}}{\widehat{A}^j_{l,m}}\right)\left(\widehat{F}^j_{l,m+1} - \widehat{U}^j_{t_{l,m+1}}\right) + \left(1 + \frac{h\widehat{A}^j_{y_{l,m}}}{\widehat{A}^j_{l,m}}\right)\left(\widehat{F}^j_{l,m-1} - \widehat{U}^j_{t_{l,m-1}}\right) + 8\left(\widehat{F}^j_{l,m} - \widehat{U}^j_{t_{l,m}}\right)\right] \\
&\quad + O(\tau^2 h^2 + \tau h^4 + h^6),
\end{aligned} \tag{35.1}$$

$$\begin{aligned}
\left[6(\delta_x^2 + \delta_y^2) + \delta_x^2\delta_y^2\right]\widehat{U}^j_{l,m} &= \frac{h^2}{2}\left[\widehat{V}^j_{t_{l+1,m}} + \widehat{V}^j_{t_{l-1,m}} + \widehat{V}^j_{t_{l,m+1}} + \widehat{V}^j_{t_{l,m-1}} + 8\widehat{V}^j_{t_{l,m}}\right] \\
&\quad + O(\tau^2 h^2 + \tau h^4 + h^6)
\end{aligned} \tag{35.2}$$

Using the above difference scheme, we find the quantities $U^j_{l,m}$ and $V^j_{l,m}$. The function values $\phi^j_{l,m}$ at the point (x_l, y_m, t_j), $l, m = 1, \ldots, N$ may be obtained by solving at each time step the Poisson equation $\nabla^2\phi = v(x, y, t)$ using the following scheme:

$$\left[6(\delta_x^2 + \delta_y^2) + \delta_x^2\delta_y^2\right]\phi^{j+1}_{l,m} = \frac{h^2}{2}\left[V^{j+1}_{l+1,m} + V^{j+1}_{l-1,m} + V^{j+1}_{l,m+1} + V^{j+1}_{l,m-1} + 8V^{j+1}_{l,m}\right] + O(h^6),\quad j = 0, 1, \ldots \tag{36}$$

To discuss the stability, we consider the following linear fourth order PDE in two space variables:

$$\frac{\partial^2\phi}{\partial t^2} + \nabla^4\phi(x,y,t) = f(x,y,t), \quad L_0 < x,y < L_1, \quad t > 0. \tag{37}$$

Applying methods (33.1)–(33.2) and (35.1)–(35.2) to the differential equation (37) results in the block tridiagonal matrix structure of the form:

$$\boldsymbol{P}\boldsymbol{y}^{j+1} = \boldsymbol{Q}\boldsymbol{y}^{j} + \boldsymbol{r}, \quad j = 0, 1, \dots \tag{38}$$

where

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_1 & \boldsymbol{P}_2 \\ -\boldsymbol{P}_2 & \boldsymbol{P}_1 \end{bmatrix}, \ \boldsymbol{Q} = \begin{bmatrix} \boldsymbol{P}_1 & -\boldsymbol{P}_2 \\ \boldsymbol{P}_2 & \boldsymbol{P}_1 \end{bmatrix}, \ \boldsymbol{y} = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix}, \ \boldsymbol{r} = \begin{bmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \end{bmatrix}.$$

The matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ are $2N^2 \times 2N^2$ block tridiagonal, $\boldsymbol{y}$ is $2N^2$ component solution vector and $\boldsymbol{r}$ denotes the $2N^2$ component column vector of known boundary and right hand side function values of the block system (38). The submatrices for the method (33.1)–(33.2) are given by

$$\boldsymbol{P}_1 = [\boldsymbol{C}, \boldsymbol{D}, \boldsymbol{C}], \ \boldsymbol{P}_2 = \frac{\lambda}{2}[\boldsymbol{E}, \boldsymbol{F}, \boldsymbol{E}]$$

where

$$\boldsymbol{C} = [0,0,0], \ \boldsymbol{D} = \boldsymbol{E} = [0,1,0], \ \boldsymbol{F} = [1,-4,1]$$

and the submatrices for the method (35.1)–(35.2) are given by

$$\boldsymbol{P}_1 = [\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{I}], \ \boldsymbol{P}_2 = \lambda[\boldsymbol{L}, \boldsymbol{M}, \boldsymbol{L}]$$

where

$$\boldsymbol{I} = [0,1,0], \ \boldsymbol{J} = [1,8,1], \ \boldsymbol{L} = [1,4,1], \ \boldsymbol{M} = [4,-20,4]$$

The eigenvalues of $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$ for method (33.1)–(33.2) are given by $\alpha_{l,m} = 1$ and $\beta_{l,m} = -2\lambda(\sin^2\sigma_l + \sin^2\zeta_m)$, for $l,m = 1,\dots,N$, respectively, where $2\sigma_l = (\pi l)/(N+1)$, $l = 1,2,\dots,N$ and $2\zeta_m = (\pi m)/(N+1)$, $m = 1,2,\dots,N$. Thus, the eigenvalues of matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ for the difference method (33.1)–(33.2) are given by $\alpha_{l,m} \pm \mathrm{i}\beta_{l,m}$, $l,m = 1,\dots,N$. The eigenvalues of $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$ for method (35.1)–(35.2) are given by

$$\eta_{l,m} = 12 - 4\sin^2\sigma_l - 4\sin^2\zeta_m, \quad l,m = 1,\dots,N,$$

and

$$\gamma_{l,m} = -\lambda\left(24\lambda\sin^2\sigma_l + 24\lambda\sin^2\zeta_m - 16\sin^2\sigma_l(1-\cos^2\zeta_m)\right), \quad l,m = 1,\dots,N,$$

respectively. Thus, the eigenvalues of matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ for the difference method (35.1)–(35.2) are given by $\eta_{l,m} \pm \mathrm{i}\gamma_{l,m}$, $l,m = 1,\dots,N$.

Hence the eigenvalues of the amplification matrix $\boldsymbol{P}^{-1}Q$ for method (33.1)–(33.2) are given by:

$$\xi_{1_{l,m}} = \frac{\alpha_{l,m} + \mathrm{i}\beta_{l,m}}{\alpha_{l,m} - \mathrm{i}\beta_{l,m}}, \quad \xi_{2_{l,m}} = \frac{\alpha_{l,m} - \mathrm{i}\beta_{l,m}}{\alpha_{l,m} + \mathrm{i}\beta_{l,m}}, \quad l,m = 1,\dots,N$$

and for the method (35.1)–(35.2):

$$\xi_{1_{l,m}} = \frac{\eta_{l,m} + \mathrm{i}\gamma_{l,m}}{\eta_{l,m} - \mathrm{i}\gamma_{l,m}}, \quad \xi_{2_{l,m}} = \frac{\eta_{l,m} - \mathrm{i}\gamma_{l,m}}{\eta_{l,m} + \mathrm{i}\gamma_{l,m}}, \quad l,m = 1,\dots,N.$$

It follows immediately that these eigenvalues are equal in modulus to unity and the proposed implicit schemes are stable for all mesh sizes.

4. Difference scheme and stability for three dimensional vibration problem

Consider the following three-space dimensional variable coefficient fourth order parabolic PDE:

$$\frac{\partial^2\phi}{\partial t^2} + A(x,y,z,t)\nabla^4\phi(x,y,z,t) = f(x,y,z,t), \quad L_0 < x,y,z < L_1, \quad t > 0, \tag{39}$$

subject to the following initial conditions:

$$\phi(x,y,z,0) = \phi_{000}(x,y,z), \quad \phi_t(x,y,z,0) = \phi_{111}(x,y,z), \quad L_0 \le x,y,z \le L_1. \tag{40}$$

Boundary conditions of second kind:

$$\phi \ \text{ and } \ \frac{\partial^2\phi}{\partial n^2} \ \text{ prescribed at the boundary}, \tag{41}$$

where n is the outward normal vector to the boundary of solution domain are given.

Let $u = \dfrac{\partial \phi}{\partial t}$, $v = \nabla^2 \phi(x, y, z, t)$, where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is three space dimensional Laplacian. Eq. (39) can now be rewritten as two simultaneous second-order PDEs of the form:

$$A(x,y,z,t)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = -\frac{\partial u}{\partial t} + f(x,y,z,t), \tag{42.1}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial v}{\partial t} \tag{42.2}$$

Let $\Omega_3 = \{(x,y,z,t) : L_0 < x, y, z < L_1,\ t > 0\}$. Using the given boundary conditions (41) and successive tangential partial derivatives, the values of $u(x, y, z, t)$ and $v(x,y,z,t)$ are known on all sides of the solution region Ω_3. In order to determine the numerical solution, we replace the solution region Ω_3 by a rectilinear mesh with step size $h > 0$ in x-, y-, z-dimensions and $\tau > 0$ in t-direction. The grid points are given by $(x_l, y_m, z_n, t_j) = (L_0 + lh, L_0 + mh, L_0 + nh, j\tau)$; $l, m, n = 0, 1, \ldots, N+1$ with $(N+1)h = L_1 - L_0$ and $j = 0, 1, \ldots$. Let $U^j_{l,m,n}$, $V^j_{l,m,n}$ be the exact solution values and $u^j_{l,m,n}$, $v^j_{l,m,n}$ be the approximate values of $u(x, y, z, t)$, $v(x,y,z,t)$, respectively at the nodal point (x_l, y_m, z_n, t_j).

Consider the following approximations, for $p, q, r = 0, \pm 1$:

$$\widehat{U}^j_{l+p,m+q,n+r} = \left(U^{j+1}_{l+p,m+q,n+r} + U^j_{l+p,m+q,n+r}\right)/2, \tag{43.1}$$

$$\widehat{U}^j_{t_{l+p,m+q,n+r}} = \left(U^{j+1}_{l+p,m+q,n+r} - U^j_{l+p,m+q,n+r}\right)/\tau, \tag{43.2}$$

$$\widehat{U}^j_{xx_{l,m,n}} = \left(\widehat{U}^j_{l+1,m,n} - 2\widehat{U}^j_{l,m,n} + \widehat{U}^j_{l-1,m,n}\right)/h^2, \tag{43.3}$$

$$\widehat{U}^j_{yy_{l,m,n}} = \left(\widehat{U}^j_{l,m+1,n} - 2\widehat{U}^j_{l,m,n} + \widehat{U}^j_{l,m-1,n}\right)/h^2, \tag{43.4}$$

$$\widehat{U}^j_{zz_{l,m,n}} = \left(\widehat{U}^j_{l,m,n+1} - 2\widehat{U}^j_{l,m,n} + \widehat{U}^j_{l,m,n-1}\right)/h^2, \tag{43.5}$$

$$\widehat{F}^j_{l+p,m+q,n+r} = f(x_{l+p}, y_{m+q}, z_{n+r}, \widehat{t}_j). \tag{43.6}$$

An $O(\tau^2 + h^2)$ method for (42.1)–(42.2) may be written as

$$\widehat{A}^j_{l,m,n}\left(\widehat{V}^j_{xx_{l,m,n}} + \widehat{V}^j_{yy_{l,m,n}} + \widehat{V}^j_{zz_{l,m,n}}\right) = -\widehat{U}^j_{t_{l,m,n}} + \widehat{F}^j_{l,m,n} + O(\tau^2 + h^2), \tag{44.1}$$

$$\widehat{U}^j_{xx_{l,m,n}} + \widehat{U}^j_{yy_{l,m,n}} + \widehat{U}^j_{zz_{l,m,n}} = \widehat{V}^j_{t_{l,m,n}} + O(\tau^2 + h^2), \tag{44.2}$$

where $\widehat{A}^j_{l,m,n} = A(x_l, y_m, z_n, \widehat{t}_j)$. The above scheme in combination with the following second order scheme is used to calculate the value of $\phi(x, y, z, t)$ at each internal grid point (x_l, y_m, z_n, t_j):

$$\begin{aligned}&\phi^{j+1}_{l+1,m,n} + \phi^{j+1}_{l-1,m,n} + \phi^{j+1}_{l,m+1,n} + \phi^{j+1}_{l,m-1,n} + \phi^{j+1}_{l,m,n+1} + \phi^{j+1}_{l,m,n-1} - 6\phi^{j+1}_{l,m,n}\\ &= h^2 V^{j+1}_{l,m,n} + O(h^4), \quad l, m, n = 1, \ldots, N, \quad j = 0, 1, \ldots\end{aligned} \tag{45}$$

An $O(\tau^2 + \tau h^2 + h^4)$ method for (42.1)–(42.2) may be written as

$$\begin{aligned}&\left[I(\delta_x^2 + \delta_y^2 + \delta_z^2) + \widehat{A}^j_{l,m,n}(\delta_x^2\delta_y^2 + \delta_y^2\delta_z^2 + \delta_z^2\delta_x^2)\right]\widehat{V}^j_{l,m,n}\\ &= \frac{h^2}{2}\Bigg[R_1\left(\widehat{F}^j_{l+1,m,n} - \widehat{U}^j_{t_{l+1,m,n}}\right) + R_2\left(\widehat{F}^j_{l-1,m,n} - \widehat{U}^j_{t_{l-1,m,n}}\right) + R_3\left(\widehat{F}^j_{l,m+1,n} - \widehat{U}^j_{t_{l,m+1,n}}\right)\\ &\quad + R_4\left(\widehat{F}^j_{l,m-1,n} - \widehat{U}^j_{t_{l,m-1,n}}\right) + R_5\left(\widehat{F}^j_{l,m,n+1} - \widehat{U}^j_{t_{l,m,n+1}}\right) + R_6\left(\widehat{F}^j_{l,m,n-1} - \widehat{U}^j_{t_{l,m,n-1}}\right)\\ &\quad + 6\left(\widehat{F}^j_{l,m,n} - \widehat{U}^j_{t_{l,m,n}}\right)\Bigg] + O(\tau^2 h^2 + \tau h^4 + h^6),\end{aligned} \tag{46.1}$$

$$\left[6(\delta_x^2+\delta_y^2+\delta_z^2)+(\delta_x^2\delta_y^2+\delta_y^2\delta_z^2+\delta_z^2\delta_x^2)\right]\widehat{U}_{l,m,n}^j$$
$$=\frac{h^2}{2}\left[\widehat{V}_{t_{l+1,m,n}}^j+\widehat{V}_{t_{l-1,m,n}}^j+\widehat{V}_{t_{l,m+1,n}}^j+\widehat{V}_{t_{l,m-1,n}}^j+\widehat{V}_{t_{l,m,n+1}}^j+\widehat{V}_{t_{l,m,n-1}}^j+6\widehat{V}_{t_{l,m,n}}^j\right]$$
$$+O(\tau^2h^2+\tau h^4+h^6), \tag{46.2}$$

where

$$I=6\widehat{A}_{l,m,n}^j-h^2\left(\frac{\widehat{A}_{x_{l,m,n}}^j}{\widehat{A}_{l,m,n}^j}\right)\widehat{A}_{x_{l,m,n}}^j-h^2\left(\frac{\widehat{A}_{y_{l,m,n}}^j}{\widehat{A}_{l,m,n}^j}\right)\widehat{A}_{y_{l,m,n}}^j-h^2\left(\frac{\widehat{A}_{z_{l,m,n}}^j}{\widehat{A}_{l,m,n}^j}\right)\widehat{A}_{z_{l,m,n}}^j$$
$$+\frac{h^2}{2}\left(\widehat{A}_{xx_{l,m,n}}^j+\widehat{A}_{yy_{l,m,n}}^j+\widehat{A}_{zz_{l,m,n}}^j\right),$$
$$R_1=\left(1-\frac{h\widehat{A}_{x_{l,m,n}}^j}{\widehat{A}_{l,m,n}^j}\right),\ R_2=\left(1+\frac{h\widehat{A}_{x_{l,m,n}}^j}{\widehat{A}_{l,m,n}^j}\right),\ R_3=\left(1-\frac{h\widehat{A}_{y_{l,m,n}}^j}{\widehat{A}_{l,m,n}^j}\right),$$
$$R_4=\left(1+\frac{h\widehat{A}_{y_{l,m,n}}^j}{\widehat{A}_{l,m,n}^j}\right),\ R_5=\left(1-\frac{h\widehat{A}_{z_{l,m,n}}^j}{\widehat{A}_{l,m,n}^j}\right),\ R_6=\left(1+\frac{h\widehat{A}_{z_{l,m,n}}^j}{\widehat{A}_{l,m,n}^j}\right).$$

Using the above difference method, we get the solutions $U_{l,m,n}^j$ and $V_{l,m,n}^j$ at the grid point (x_l, y_m, z_n, t_j). The solution values $\phi(x, y, z, t)$ at the points (x_l, y_m, z_n, t_j), $l, m, n = 1, \dots, N$, $j = 1, 2, \dots$ may be obtained by solving at each time step the Poisson equation $\nabla^2\phi = v(x, y, z, t)$ using the following scheme:

$$\left[6(\delta_x^2+\delta_y^2+\delta_z^2)+(\delta_x^2\delta_y^2+\delta_y^2\delta_z^2+\delta_z^2\delta_x^2)\right]\phi_{l,m,n}^{j+1}$$
$$=\frac{h^2}{2}\left[V_{l+1,m,n}^{j+1}+V_{l-1,m,n}^{j+1}+V_{l,m+1,n}^{j+1}+V_{l,m-1,n}^{j+1}+V_{l,m,n+1}^{j+1}+V_{l,m,n-1}^{j+1}+6V_{l,m,n}^{j+1}\right]+O(h^6),$$
$$l,m,n=1,\dots,N,\ \ j=0,1,\dots. \tag{47}$$

To discuss the stability, we consider the following linear fourth order PDE in three space variables:

$$\frac{\partial^2\phi}{\partial t^2}+\nabla^4\phi(x,y,z,t)=f(x,y,z,t),\quad L_0<x,y,z<L_1,\quad t>0 \tag{48}$$

The methods (44.1)–(44.2) and (46.1)–(46.2) when applied to the above equation leads to the following system of equations:

$$u_{l,m,n}^{j+1}+\frac{\lambda}{2}(\delta_x^2+\delta_y^2+\delta_z^2)v_{l,m,n}^{j+1}=u_{l,m,n}^j-\frac{\lambda}{2}(\delta_x^2+\delta_y^2+\delta_z^2)v_{l,m,n}^j+k\widehat{F}_{l,m,n}^j, \tag{49.1}$$

$$-\frac{\lambda}{2}(\delta_x^2+\delta_y^2+\delta_z^2)u_{l,m,n}^{j+1}+v_{l,m,n}^{j+1}=\frac{\lambda}{2}(\delta_x^2+\delta_y^2+\delta_z^2)u_{l,m,n}^j+v_{l,m,n}^j, \tag{49.2}$$

and

$$(12+\delta_x^2+\delta_y^2+\delta_z^2)u_{l,m,n}^{j+1}+\lambda(6\delta_x^2+6\delta_y^2+6\delta_z^2+\delta_x^2\delta_y^2+\delta_y^2\delta_z^2+\delta_z^2\delta_x^2)v_{l,m,n}^{j+1}$$
$$=(12+\delta_x^2+\delta_y^2+\delta_z^2)u_{l,m,n}^j+\lambda(6\delta_x^2+6\delta_y^2+6\delta_z^2+\delta_x^2\delta_y^2+\delta_y^2\delta_z^2+\delta_z^2\delta_x^2)v_{l,m,n}^j$$
$$+k(12+\delta_x^2+\delta_y^2+\delta_z^2)\widehat{F}_{l,m,n}^j, \tag{50.1}$$

$$-\lambda(6\delta_x^2+6\delta_y^2+6\delta_z^2+\delta_x^2\delta_y^2+\delta_y^2\delta_z^2+\delta_z^2\delta_x^2)u_{l,m,n}^{j+1}+(12+\delta_x^2+\delta_y^2+\delta_z^2)v_{l,m,n}^{j+1}$$
$$=\lambda(6\delta_x^2+6\delta_y^2+6\delta_z^2+\delta_x^2\delta_y^2+\delta_y^2\delta_z^2+\delta_z^2\delta_x^2)u_{l,m,n}^j+(12+\delta_x^2+\delta_y^2+\delta_z^2)v_{l,m,n}^j, \tag{50.2}$$

respectively, which results in the block tridiagonal matrix structure of the form:

$$\boldsymbol{S}\boldsymbol{y}^{j+1}=\boldsymbol{T}\boldsymbol{y}^j+\boldsymbol{d},\quad j=0,1,\dots \tag{51}$$

where

$$\boldsymbol{S}=\begin{bmatrix}\boldsymbol{S}_1 & \boldsymbol{S}_2\\ -\boldsymbol{S}_2 & \boldsymbol{S}_1\end{bmatrix},\ \boldsymbol{T}=\begin{bmatrix}\boldsymbol{S}_1 & -\boldsymbol{S}_2\\ \boldsymbol{S}_2 & \boldsymbol{S}_1\end{bmatrix},\ \boldsymbol{y}=\begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{v}\end{bmatrix},\ \boldsymbol{d}=\begin{bmatrix}\boldsymbol{d}_1\\ \boldsymbol{d}_2\end{bmatrix}$$

The matrices $\boldsymbol{S}$ and $\boldsymbol{T}$ are $2N^3 \times 2N^3$ block tridiagonal, $\boldsymbol{y}$ is $2N^3$ component solution vector and $\boldsymbol{d}$ denotes the $2N^3$ component column vector of known boundary values and right hand side function values of the block system (51).

The eigenvalues of $\boldsymbol{S_1}$ and $\boldsymbol{S_2}$ for method (49.1)–(49.2) are given by $\alpha_{l,m,n} = 1$ and $\beta_{l,m,n} = -2\lambda(\sin^2\sigma_l + \sin^2\zeta_m + \sin^2\omega_n)$, for $l, m, n = 1, \dots, N$, respectively, where $2\sigma_l = (\pi l)/(N + 1)$, $l = 1, 2, \dots, N$, $2\zeta_m = (\pi m)/(N + 1)$, $m = 1, 2, \dots, N$

Table 1
Observed absolute errors by the proposed $O(\tau^2+\tau h^2+h^4)$ algorithm for Example 1 with different values of the parameters $N+1$ and τ at various time intervals.

Methods	t	$N+1$	τ	$x=0.1$	$x=0.2$	$x=0.3$	$x=0.4$	$x=0.5$
Proposed method	0.02	20	0.00125	8.0105(−07)	1.5237(−06)	2.0972(−06)	2.4654(−06)	2.5922(−06)
		40		5.1080(−08)	9.7161(−08)	1.3373(−07)	1.5721(−07)	1.6531(−07)
CPU time (s)	0.02	20	0.00125	0.0191	0.0183	0.0183	0.0184	0.0186
		40		0.0356	0.0315	0.0322	0.0308	0.0310
Proposed method	0.05	20	0.005	9.9372(−07)	1.8902(−06)	2.6015(−06)	3.0583(−06)	3.2157(−06)
		40		1.7033(−07)	3.2399(−07)	4.4594(−07)	5.2424(−07)	5.5122(−07)
CPU time (s)	0.05	20	0.005	0.0241	0.0248	0.0242	0.0244	0.0246
		40		0.1496	0.1483	0.1489	0.1512	0.1513
Ref. [21]	0.02	181	0.005	1.50(−07)	2.90(−07)	3.90(−07)	4.60(−07)	4.90(−07)
	0.05			1.10(−06)	2.09(−06)	2.88(−06)	3.38(−06)	3.56(−06)
Ref. [19]	0.02	121	0.005	4.80(−06)	9.70(−06)	1.40(−05)	1.90(−05)	2.40(−05)
		191		5.20(−06)	2.10(−06)	3.10(−06)	4.20(−06)	5.20(−06)
		521		4.90(−07)	9.90(−07)	1.40(−06)	1.90(−06)	2.40(−06)
Ref. [17]	0.02	20	0.00125	9.20(−06)	7.90(−06)	2.80(−06)	9.80(−07)	2.50(−06)
	0.05		0.005	9.30(−06)	8.00(−06)	2.80(−06)	1.00(−06)	2.70(−06)
Ref. [18]	0.02	20	0.00125	9.07(−06)	7.79(−06)	2.75(−06)	1.01(−06)	2.59(−06)
	0.05		0.005	1.87(−06)	2.13(−05)	1.49(−05)	8.60(−06)	5.96(−06)

and $2\omega_n=(\pi n)/(N+1),\ n=1,2,\ldots,N$. Thus, the eigenvalues of matrices $\boldsymbol{S}$ and $\boldsymbol{T}$ for the difference method (49.1)–(49.2) are given by $\alpha_{l,m,n}\pm \mathrm{i}\beta_{l,m,n},\ l,m,n=1,\ldots,N$.

The eigenvalues of $\boldsymbol{S_1}$ and $\boldsymbol{S_2}$ for method (50.1)–(50.2) are given by

$$\eta_{l,m,n}=12-4\sin^2\sigma_l-4\sin^2\zeta_m-4\sin^2\omega_n,\quad l,m,n=1,\ldots,N,$$

and

$$\gamma_{l,m,n}=-\lambda\Big[24\big(\sin^2\sigma_l+\sin^2\zeta_m+\sin^2\omega_n\big)-16\big(\sin^2\sigma_l(1-\cos^2\zeta_m)+\sin^2\zeta_m(1-\cos^2\omega_n)+\sin^2\omega_n(1-\cos^2\sigma_l)\big)\Big],\quad l,m,n=1,\ldots,N,$$

respectively. Thus, the eigenvalues of matrices $\boldsymbol{S}$ and $\boldsymbol{T}$ for the difference method (50.1)–(50.2) are given by $\eta_{l,m,n}\pm \mathrm{i}\gamma_{l,m,n}$, $l,m,n=1,\ldots,N$.

The eigenvalues of the amplification matrix $\boldsymbol{S}^{-1}T$ for method (49.1)–(49.2) are:

$$\xi_{1_{l,m,n}}=\frac{\alpha_{l,m,n}+\mathrm{i}\beta_{l,m,n}}{\alpha_{l,m,n}-\mathrm{i}\beta_{l,m,n}},\quad \xi_{2_{l,m,n}}=\frac{\alpha_{l,m,n}-\mathrm{i}\beta_{l,m,n}}{\alpha_{l,m,n}+\mathrm{i}\beta_{l,m,n}},\quad l,m,n=1,\ldots,N$$

and for the method (50.1)–(50.2):

$$\xi_{1_{l,m,n}}=\frac{\eta_{l,m,n}+\mathrm{i}\gamma_{l,m,n}}{\eta_{l,m,n}-\mathrm{i}\gamma_{l,m,n}},\quad \xi_{2_{l,m,n}}=\frac{\eta_{l,m,n}-\mathrm{i}\gamma_{l,m,n}}{\eta_{l,m,n}+\mathrm{i}\gamma_{l,m,n}},\quad l,m,n=1,\ldots,N,$$

which are equal in modulus to one and hence the proposed schemes are unconditionally stable.

5. Numerical examples

In this section, we present numerical results to validate our theoretical considerations. The matrices represented by the proposed formulas are block tridiagonal. Gauss–Seidel iteration method has been used for solving coupled system of equations [31,32] and in each case the iterations are terminated once the absolute error tolerance 10^{-12} has been reached. For all the tests, the used CPU time has been reported in the tables.

Example 1. In order to compare our method with the existing techniques, we consider homogenous beam equation (14) with $EI=\rho=A=1$ having the form [17–19,21]:

$$\frac{\partial^2\phi}{\partial t^2}+\frac{\partial^4\phi}{\partial x^4}=(\pi^4-1)\sin\pi x\cos t,\quad 0<x<1,\ \ t>0 \tag{52}$$

with the exact solution

$$\phi(x,t)=\sin\pi x\cos t$$

We applied the proposed $O(\tau^2+\tau h^2+h^4)$ algorithm to this problem with $h=0.05, 0.025$ and $\tau=0.00125, 0.005$. The computed solutions are compared with the exact solutions and the absolute errors at points $x=0.1, 0.2, 0.3, 0.4, 0.5$ are reported in Table 1 at time levels $t=0.02$ and $t=0.05$ using 16 and 10 time steps, respectively. The approximate solutions obtained

Table 2
Maximum absolute relative errors for Example 2 at $t = 0.01$ for various values of $\lambda = \tau/h^2$.

λ	h	τ	Proposed $O(\tau^2+\tau h^2+h^4)$-method	CPU time (s)	$O(\tau^2+h^4)$-method discussed in [20]	$O(\tau^4+h^4)$-method discussed in [20]	Method discussed in [12]	Method discussed in [11]
0.05	0.05	0.000125	2.9321(−11)	0.0214	5.21(−08)	5.33(−08)	9.90(−08)	1.90(−06)
0.1		0.00025	4.0843(−12)	0.0189	1.03(−07)	9.97(−08)	8.10(−08)	7.20(−07)
0.25		0.000625	3.7383(−12)	0.0169	3.74(−08)	3.51(−08)	6.90(−08)	4.10(−07)

Table 3
Observed absolute errors by the proposed algorithm for Example 3 with different values of the parameters $N+1$ and τ at various time intervals.

Methods	t	h	τ	$x = 0.1$	$x = 0.2$	$x = 0.3$	$x = 0.4$	$x = 0.5$
Proposed $O(\tau^2+\tau h^2+h^4)$	0.02	0.05	0.00125	2.0025(−10)	3.3994(−10)	7.5562(−10)	1.0342(−09)	1.2053(−09)
method	0.05		0.005	6.2981(−09)	3.9246(−09)	4.7315(−09)	4.9752(−09)	5.2624(−09)
CPU time (s)	0.02	0.05	0.00125	0.0409	0.0425	0.0411	0.0395	0.0409
	0.05		0.005	0.0322	0.0305	0.0327	0.0332	0.0313
$O(\tau^2+h^4)$ method in [20]	0.02	0.05	0.00125	8.63(−08)	3.59(−08)	6.29(−08)	1.63(−08)	2.97(−08)
	0.05		0.005	8.56(−08)	5.14(−08)	9.03(−08)	2.39(−08)	4.61(−08)
$O(\tau^4+h^4)$ method in [20]	0.02	0.05	0.00125	8.42(−08)	2.62(−08)	5.32(−08)	1.45(−08)	2.89(−08)
	0.05		0.005	8.35(−08)	4.51(−08)	8.25(−08)	2.33(−08)	4.52(−08)

are also compared with the results of [17–19,21]. It is observed that the present approach uses less spatial grid points and still gives smaller absolute errors than the methods employed in the previous work.

Example 2. Consider the following homogenous fourth order parabolic equation:

$$\frac{\partial^2\phi}{\partial t^2} + \left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4\phi}{\partial x^4} = 0, \quad \frac{1}{2} < x < 1, \quad t > 0 \tag{53}$$

The exact solution for this problem is

$$\phi(x,t) = \left(1 + \frac{x^5}{120}\right)\sin t$$

The approximate solutions are computed at $t = 0.01$ with $h = 0.05$ and $\tau = 0.000125, 0.00025, 0.000625$ over 80, 40 and 16 time steps, respectively using proposed $O(\tau^2+\tau h^2+h^4)$ method. The maximum absolute relative errors defined by

$$\max\left|\frac{\phi^*(x_l,t) - \phi(x_l,t)}{\phi^*(x_l,t)}\right|, \qquad l = 1,\ldots,N,$$

where $\phi(x_l,\ t)$, $\phi^*(x_l,\ t)$ denote the numerical and exact solutions, respectively are presented in Table 2 at $t = 0.01$. The proposed method produces more accurate results in comparison with the existing methods of same accuracy $O(\tau^2+h^4)$.

Example 3. Consider the following homogenous fourth order parabolic equation with variable coefficient:

$$\frac{\partial^2\phi}{\partial t^2} + \left(\frac{x}{\sin x} - 1\right)\frac{\partial^4\phi}{\partial x^4} = 0, \quad 0 < x < 1, \quad t > 0 \tag{54}$$

The exact solution for this problem is

$$\phi(x,t) = (x - \sin x)e^{-t}$$

The absolute errors using the $O(\tau^2+\tau h^2+h^4)$ scheme at points $x = 0.1, 0.2, 0.3, 0.4, 0.5$ are given in Table 3 at time levels $t = 0.02$ and $t = 0.05$ using 16 and 10 time steps, respectively with $h = 0.05$, $\tau = 0.00125$ and $\tau = 0.005$, respectively. The results are compared with the methods discussed in [20] and are found to be much more accurate. We present the two-dimensional visual comparison between numerical and exact solutions at $t = 1, 2, 3$ and 4 for $\lambda = 1.6$ using the proposed $O(\tau^2+\tau h^2+h^4)$ method in Fig. 1.

Example 4. Consider the following Euler–Bernoulli beam equation with the exact solution $\phi(x,t) = (\sin^3\pi x + x)t^2e^{-t}$:

$$\frac{\partial^2\phi}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left((1+\sin\pi x)\frac{\partial^2\phi}{\partial x^2}\right) = f(x,t), \quad 0 < x < 1, \quad t > 0, \tag{55}$$

and $f(x,\ t)$ is consistent with the exact solution. The maximum absolute errors (MAEs) at $t = 1, 4$ and 8 are tabulated in Table 4 for a fixed value of the mesh parameter $\lambda = 1.6$ with $N+1 = 8, 16, 32$ and 64 using the derived $O(\tau^2+\tau h^2+h^4)$

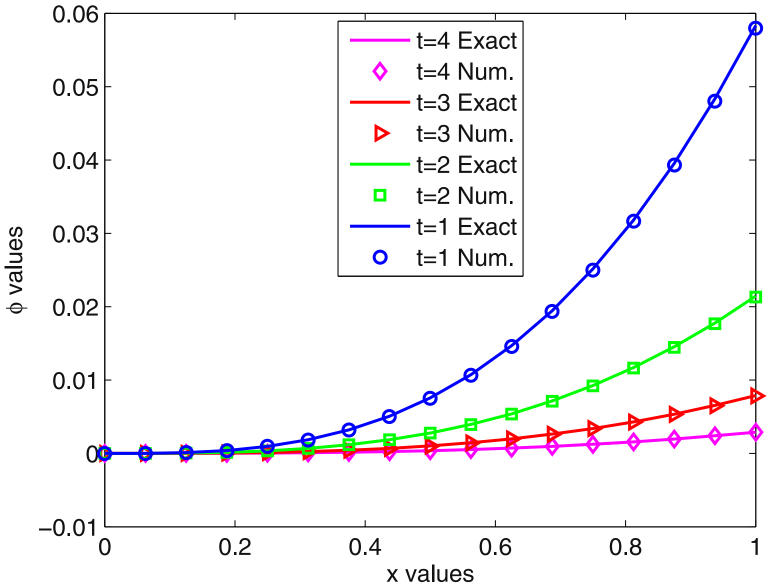

Fig. 1. Example 3: Comparison between numerical and exact solutions of Eq. (54) at different times.

Table 4
MAEs and order of convergence for Example 4 at $t = 1, 4$ and 8 using $O(\tau^2 + \tau h^2 + h^4)$ method.

h		$t = 1$		$t = 4$		$t = 8$	
		MAE	Order	MAE	Order	MAE	Order
1/8	ϕ	8.7717(−03)	–	6.7337(−03)	–	5.9814(−04)	–
	ϕ_{xx}	2.7453(−01)	–	2.0857(−01)	–	1.9327(−02)	–
CPU time (s)		0.0522		0.1592		0.3057	
1/16	ϕ	5.1038(−04)	4.1032	3.9594(−04)	4.0880	3.5996(−05)	4.0546
	ϕ_{xx}	1.5713(−02)	4.1269	1.2402(−02)	4.0719	7.9348(−04)	4.6063
CPU time (s)		0.6767		2.9539		5.7006	
1/32	ϕ	3.1560(−05)	4.0154	2.4426(−05)	4.0188	2.2228(−06)	4.0174
	ϕ_{xx}	1.0105(−03)	3.9588	7.8777(−04)	3.9767	5.7687(−05)	3.7819
CPU time (s)		13.9656		60.5975		117.4496	

method. For each spatial step size h, the corresponding time step size is chosen as $\tau \propto h^2$. With this choice of time step size, the theoretical order of convergence becomes $O(h^4)$, i.e., the method is fourth order accurate in space. The order of convergence is numerically estimated equal to four in Table 4. Fig. 2 shows the exact and numerical results at different time levels.

Example 5 (Non-uniform beam)**.** Consider a simply supported non-uniform beam of length $L = 1$ m whose motion is governed by the PDE (1). The cross-sectional area $A(x)$ and moment of inertia $I(x)$ vary according to the relations [33,34]:

$$A(x) = A_0(1 + \alpha x)^4, \quad I(x) = I_0(1 + \alpha x)^4$$

where A_0 and I_0 are the cross-sectional area and the moment of inertia at $x = 0$, respectively and α is a positive constant. Specifically, it is assumed that the cross-section is rectangular at $x = 0$, $b_0 = 0.25$ m be the width and $h_0 = 0.25$ m be the height, so $A_0 = b_0 h_0$ and $I_0 = (b_0 h_0^3)/12$. The material properties are chosen to be [34]: $E = 2.48 \times 10^7$ kNm^{-2}, the Young's modulus and $\rho = 2.4028 \times 10^3$ kg m^{-3}, the mass density. In this test, we take $\phi(x, t) = \sinh t \sin \pi x$ to be the exact solution of Eq. (1) and the homogenous function $f(x, t)$ is obtained using this exact solution. Table 5 gives the MAEs and order of convergence computed by the proposed difference method (8.1)–(8.2) for $\lambda = 1.6$ with different meshes $N + 1 = 8, 16$ and 32 at $t = 1$ for various values of α.

Example 6. Consider a beam with exponentially varying width and constant height yielding variable cross-section resting on elastic foundation undergoing transverse vibration under axial tensile force described by the PDE (1). The width of the beam varies over length coordinates as follows:

$$b(x) = b_0 e^{-\alpha x}$$

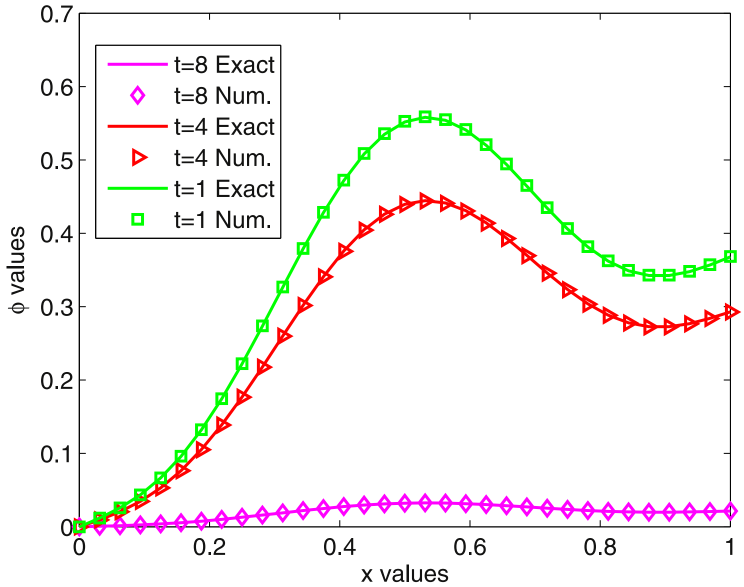

Fig. 2. Example 4: Comparison between numerical and exact solutions of Eq. (55) at different times.

Table 5
MAEs and order of convergence for non-uniform beam, Example 5 at $t = 1$ using $O(\tau^2 + \tau h^2 + h^4)$ method (8.1)–(8.2).

h	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 1.0$	
	MAE	Order	MAE	Order	MAE	Order
1/8	1.4304(−04)	–	1.4685(−04)	–	4.6640(−04)	–
CPU time (s)	0.0501		0.0493		0.0506	
1/16	8.8739(−06)	4.0107	9.1107(−06)	4.0106	3.0473(−05)	3.9360
CPU time (s)	0.5837		0.7051		0.6333	
1/32	5.5953(−07)	3.9873	5.6659(−07)	4.0072	1.9431(−06)	3.9711
CPU time (s)	10.3982		10.2332		10.7831	

Table 6
MAEs and order of convergence for Example 6 at $t = 1$ using $O(\tau^2 + \tau h^2 + h^4)$ method (8.1)–(8.2).

h	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 1.0$	
	MAE	Order	MAE	Order	MAE	Order
1/8	1.4127(−04)	–	1.2922(−04)	–	9.7592(−05)	–
CPU time (s)	0.0493		0.0496		0.0508	
1/16	8.7561(−06)	4.0120	8.0218(−06)	4.0098	6.0793(−06)	4.0048
CPU time (s)	0.6014		0.5978		0.5737	
1/32	5.5180(−07)	3.9881	5.0698(−07)	3.9839	3.8024(−07)	3.9989
CPU time (s)	10.5919		10.3191		9.9303	

Here b_0 is the width in the left end of beam and α is the non-uniformity parameter. When width of the beam varies over the beam length as above, then cross-section and moment of inertia varies as follows:

$$A(x) = A_0 e^{-\alpha x}, \quad I(x) = I_0 e^{-\alpha x},$$

where A_0 and I_0 are cross-section and moment of inertia in the left end of beam, respectively. The right hand function $f(x, t)$ in (1) is determined by the exact solution $\phi(x,t) = \sinh t \sin \pi x$. We consider the same physical parameters as chosen in Example 5. In Table 6, we report MAEs and order of convergence of the numerical solution obtained by applying the method (8.1)–(8.2) at $t = 1$ with $\lambda = 1.6$.

Example 7. We consider the one-dimensional fourth order parabolic PDE with variable coefficient of the form:

$$\frac{\partial^2 \phi}{\partial t^2} + (1 + x^2 + t^2)\frac{\partial^4 \phi}{\partial x^4} = f(x,t), \quad 0 < x < 1, \quad t > 0 \tag{56}$$

Table 7
MAEs and order of convergence for Example 7 at $t = 1$.

h		Proposed $O(\tau^2+\tau h^2+h^4)$-method (19.1)–(19.2)		$O(\tau^2+h^2)$-method (17.1)–(17.2)	
		MAE	Order	MAE	Order
1/8	ϕ	3.3124(−05)	–	1.3953(−03)	–
	ϕ_{xx}	1.9091(−04)	–	6.0472(−02)	–
CPU time (s)		0.0458		0.04233	
1/16	ϕ	2.1242(−06)	3.9629	4.1623(−04)	1.7451
	ϕ_{xx}	1.1946(−05)	3.9983	1.5749(−02)	1.9410
CPU time (s)		0.5842		0.6565	
1/32	ϕ	1.3451(−07)	3.9811	1.0413(−04)	1.9990
	ϕ_{xx}	8.7869(−07)	3.7650	3.9526(−03)	1.9944
CPU time (s)		13.4579		15.7557	

Table 8
MAEs and order of convergence for Example 8 at $t = 1$.

h		Proposed $O(\tau^2+\tau h^2+h^4)$-method (35.1)–(35.2)		$O(\tau^2+h^2)$-method (33.1)–(33.2)	
		MAE	Order	MAE	Order
1/8	ϕ	3.3459(−04)	–	3.1527(−02)	–
	$\nabla^2\phi$	5.4085(−03)	–	3.1782(−01)	–
CPU time (s)		0.3253		0.4847	
1/16	ϕ	2.0708(−05)	4.0141	7.7747(−03)	2.0197
	$\nabla^2\phi$	3.3910(−04)	3.9954	7.8554(−02)	2.0165
CPU time (s)		12.6545		16.7472	
1/32	ϕ	1.3020(−06)	3.9914	1.9356(−03)	2.0060
	$\nabla^2\phi$	2.1166(−06)	4.0019	1.9557(−02)	2.0060
CPU time (s)		549.9453		810.6188	

where the $f(x,t) = \left[1+\pi^4\left(1+x^2+t^2\right)\right]e^{-t}\sin\pi x$ and the exact solution is $\phi(x,t) = e^{-t}\sin\pi x$. The MAEs and order of convergence for $\phi(x, t)$ and $\phi_{xx}(x, t)$ using the difference schemes (19.1)–(19.2) and (17.1)–(17.2) are tabulated in Table 7 at $t = 1$ for $\lambda = 1.6$ with $N+1 = 8, 16$ and 32.

Example 8. We seek the numerical solution of the following variable coefficient two dimensional fourth order parabolic PDE with the exact solution $\phi(x,y,t) = \sinh t\sin\pi x\sin\pi y$:

$$\frac{\partial^2\phi}{\partial t^2} + (1+x^2+y^2)\nabla^4\phi(x,y,t) = f(x,y,t), \quad 0<x,y<1, \quad t>0 \tag{57}$$

where

$$f(x,y,t) = \left(1+4\pi^4(1+x^2+y^2)\right)\sinh t\sin\pi x\sin\pi y.$$

The MAEs and numerical order of convergence for $\phi(x, y, t)$ and spatial Laplacian $\nabla^2\phi(x, y, t)$ are reported in Table 8 for a fixed $\lambda = 1.6$ with $N+1 = 8, 16$ and 32 at $t = 1.0$.

Example 9. Consider the following three dimensional fourth order parabolic PDE with the exact solution $\phi(x,y,z,t) = \sinh t\sin\pi x\sin\pi y\sin\pi z$:

$$\frac{\partial^2\phi}{\partial t^2} + \nabla^4\phi(x,y,z,t) = f(x,y,z,t), \quad 0<x,y,z<1, \quad t>0 \tag{58}$$

where

$$f(x,y,z,t) = \left(1+9\pi^4\right)\sinh t\sin\pi x\sin\pi y\sin\pi z.$$

The MAEs and numerical order of convergence for $\phi(x, y, z, t)$ and spatial Laplacian $\nabla^2\phi(x, y, z, t)$ are reported in Table 9 for a fixed $\lambda = 1.6$ with $N+1 = 4, 8$ and 16 at $t = 0.5$ using proposed $O(\tau^2+\tau h^2+h^4)$ and central difference $O(\tau^2+h^2)$ methods. In Fig. 3, we display in a Log/Log scale, the MAEs in $\phi(x, y, z, t)$ and $\nabla^2\phi(x, y, z, t)$ at $t = 0.5$. It is observed from Fig. 3 that the slope of the graph is almost constant around four for the $O(\tau^2+\tau h^2+h^4)$ method.

6. Final discussion

The available methods for the numerical solution of one and two space dimensional vibration problem were of low order, three-level in nature and computationally not very efficient due to stability restriction. In this article, we have developed two-level implicit difference schemes of order two in time and four in space for solving multi-space dimensional

Table 9
MAEs and order of convergence for Example 9 at $t = 0.5$.

h		Proposed $O(\tau^2 + \tau h^2 + h^4)$-method (50.1)–(50.2)		$O(\tau^2 + h^2)$-method (49.1)–(49.2)	
		MAE	Order	MAE	Order
1/4	ϕ	4.8522(−03)	–	5.6320(−02)	–
	$\nabla^2\phi$	8.3259(−02)	–	8.0660(−01)	–
CPU time (s)		0.0665		0.0575	
1/8	ϕ	2.6488(−04)	4.1952	1.2648(−02)	2.1547
	$\nabla^2\phi$	4.2201(−03)	4.3023	1.7243(−01)	2.2258
CPU time (s)		1.6781		1.3281	
1/16	ϕ	1.6627(−05)	3.9937	3.1712(−03)	1.9958
	$\nabla^2\phi$	2.6846(−04)	3.9745	4.4089(−02)	1.9675
CPU time (s)		173.1176		159.8605	

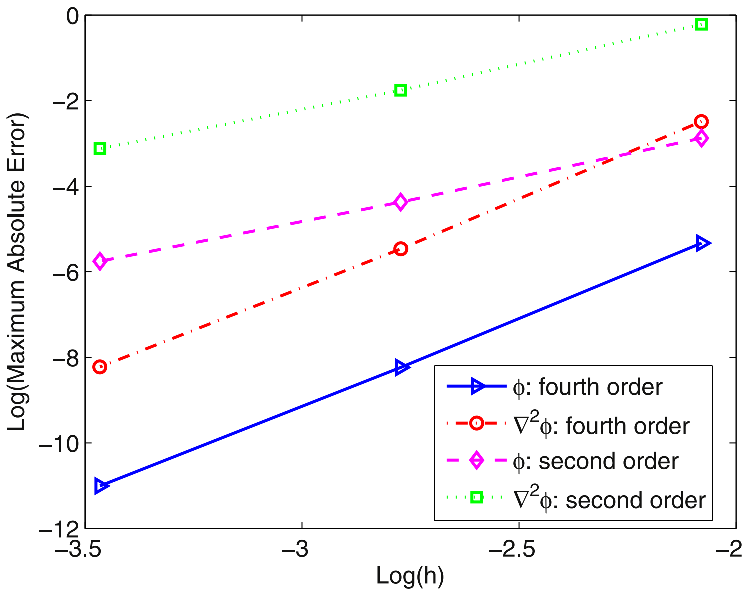

Fig. 3. Example 9: Error plot at $t = 0.5$ for $\lambda = 1.6$.

vibration problems. The proposed difference schemes utilize only three, nine and nineteen spatial grid points for one, two and three dimensional fourth order parabolic equations, respectively of a single compact stencil. One advantage of this formulation is that the boundary conditions are incorporated in a natural way without the use of any extra nodes or special schemes adjacent to the boundary, thereby eliminating the usual complexity encountered with the central difference methods. In addition, the value of the space Laplacian $\nabla^2\phi$ which is often of interest in some applied problems is computed as a by-product of the methods. The proposed algorithm has been tested for non-uniform beams which provide better distribution of mass and strength than uniform beams and are useful in various engineering applications. Numerical results demonstrate the accuracy and efficiency of the derived scheme for non-uniform Euler–Bernoulli beams. The proposed two-level finite difference methods are shown to be unconditionally stable for model fourth order parabolic equations in one, two and three space dimensions. The numerical results clearly indicate that the proposed method produce better results in comparison with the existing methods of [11,12,17–21] for the one dimensional beam equation. Also, the new fourth order formulas produce much more accurate results on relatively coarser grids than the standard second order methods. It is exhibited from Tables 4–9 that the proposed discretization produces fourth-order accurate results for a fixed mesh ratio parameter λ, i.e., when $\tau \propto h^2$. The methods are easily implemented, concise, accurate and computationally attractive for multi-space dimensional fourth order parabolic PDEs with variable coefficients. The approach presented in this paper will be helpful for the development of stable algorithms for other fourth order time dependent problems.

Acknowledgments

The authors would like to thank the anonymous referees for their constructive comments and remarks which helped improve the quality of the paper.

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