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MDS and I-perfect poset block codes

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ABSTRACT

We obtain the Singleton bound for poset block codes and define a maximum distance separable poset block code (MDS (P, π)-code) as a code meeting this bound. We extend the concept of I-balls to poset block metric and describe r-perfect and MDS (P, π)-codes in terms of I-perfect codes. As a special case when all the blocks have the same dimension, we establish that MDS (P, π)-codes are same as I-perfect codes for $I \in \mathcal{I}^{\frac{n-k}{m}}(P)$. We show that the duality result also holds for this case. Further, we determine the weight enumerator of an MDS (P, π)-code.

1. Introduction

One of the important aspects of coding theory is to construct codes capable of correcting various errors that occur frequently in communication channels. Since the error correction capability of a code is best determined in terms of its minimum distance,

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therefore a code with larger minimum distance has better error-correction capability. Most of the studies have concentrated on linear codes endowed with the Hamming metric. For a linear code $C \subseteq \mathbb{F}_q^n$ of dimension k and minimum distance d_H with respect to the Hamming metric, the Singleton bound [13] is

$$d_H \leq n - k + 1.$$

Codes that achieve equality in the Singleton bound are called maximum distance separable (MDS) codes. For given n and k, MDS codes have the largest minimum distance and hence the best error-correction capabilities. The class of MDS codes, therefore, has been of utmost interest in coding theory. MDS codes have applications in finite geometries and in combinatorics as well.

The study of codes endowed with a metric other than the Hamming metric gained momentum towards 1990's with the introduction of poset metric by Brualdi et al. [2]. The concept of poset metric was motivated by Niederreiter's generalization of a classical problem of coding theory [9,10]. Codes endowed with poset metric, namely the Rosenbloom-Tsfasman metric (RT-metric) [12], are of practical significance in case of interference in several consecutive channels. In [3,14], the authors proved the Singleton bound for RT-metric and generalized the notion of MDS codes. Extending their approach, Hyun and Kim [5] studied MDS poset codes and introduced the concept of I-perfect codes by extending the concept of elementary box defined for RT-metric. Further, it was proved that the duality result holds for MDS poset codes.

Feng et al. in [4] introduced block metric and studied MDS block codes. As seen in [4], the dual of an MDS block code need not be MDS unless all the blocks have the same dimension. Poset block metric was introduced by Alves et al. in [1] unifying poset metric and block metric. Codes endowed with the Niederreiter-Rosenbloom-Tsfasman block metric (NRT block metric), a particular case of poset block metric when the poset is a chain, have been classified in [11] based on their generalized weight hierarchy. The authors have developed much of the classical theory for NRT block codes by obtaining a canonical representative for each such class.

The present paper aims to introduce MDS poset block codes and extend the concept of I-perfect codes to the case of poset block metric. In section 3, we present a result relating the (P, π)-weight of a codeword and a parity check matrix of the code in terms of an ideal of the poset P. We prove Singleton bound for an (n, k, d) poset block code and define an MDS (P, π)-code. In section 4, we extend the concept of I-perfect codes to the case of poset block metric and show that they are systematical. We relate an I-perfect code with MDS (P, π)-code and r-perfect code. We also prove the duality result for I-perfect codes. In section 5, we prove that when all the blocks have the same dimension, codewords of an MDS (P, π)-code are uniformly distributed among the I-balls and vice-versa. The duality result also holds in this case. Further, we completely determine the weight distribution of an MDS (P, π)-code when all the blocks have the same dimension as in the classical case. Lastly, we show that MDS poset block codes are in abundance.

2. Preliminaries

In this section, we give basic definitions and results of a poset block metric and additive characters to be used in the subsequent sections.

Let $P = ([s], \leq)$ denote a **poset** with s elements, where $[s] := \{1, 2, \dots, s\}$ and $\leq$ is a partial order on $[s]$. An **ideal** I of the poset P is a subset of $[s]$ such that if $i \in I$ and $j \leq i$, then $j \in I$. For $X \subseteq [s]$, we denote by $\langle X \rangle$ the smallest ideal containing X, called the **ideal generated by** X. We prefer to denote the ideal generated by $X = \{i\}$ as $\langle i \rangle$ rather than $\langle \{i\} \rangle$ as in [1]. The t**-th level set** $\Gamma^{(t)}(P)$ of P is defined as

$$\Gamma^{(t)}(P) = \{i \in P : |\langle i \rangle| = t\},$$

where $|\cdot|$ denotes the cardinality of the given set. We denote by $\mathcal{I}(P)$ the set of all ideals of P and by $\mathcal{I}^r(P)$ the set of all ideals of P of cardinality r. We use the notation $M(I)$ to denote the set of all maximal elements of I and I_M to denote $I \setminus M(I)$.

Proposition 2.1. [5, Proposition 1.1]

1. *Let $0 \leq r \leq r' \leq n$ and $I \in \mathcal{I}^r(P)$. Then there exists $J \in \mathcal{I}^{r'}(P)$ such that $I \subseteq J$.*
2. *Let $0 \leq r' \leq r \leq n$ and $I \in \mathcal{I}^r(P)$. Then there exists $J \in \mathcal{I}^{r'}(P)$ such that $J \subseteq I$.*

For a given poset P, we define the **dual poset** $\widetilde{P}$ as follows:

P and $\widetilde{P}$ have the same underlying set and $i \leq j$ in P if and only if $j \leq i$ in $\widetilde{P}$.

Proposition 2.2. [5, Lemma 1.2] *Let P be a poset on $[s]$ and $\widetilde{P}$ be its dual poset. Then the ideals of $\widetilde{P}$ are precisely the complements of the ideals of P, that is, $\mathcal{I}(\widetilde{P}) = \{I^c : I \in \mathcal{I}(P)\}$, where I^c is equal to $[s] \setminus I$.*

Let P and Q be posets on the same underlying set. We say that Q is **finer** than P if $i \leq j$ in P implies that $i \leq j$ in Q. We say a poset $P = ([s], \leq)$ is a **hierarchical poset** if there is a partition

$$[s] = \bigsqcup_{\ell=1,2,\dots,t} H_\ell$$

of $[s]$ such that $i \leq j$ if and only if $i = j$ or $i \in H_{\ell_i}$, $j \in H_{\ell_j}$ and $\ell_i < \ell_j$. Let $s_\ell = |H_\ell|$ for $1 \leq \ell \leq t$. Then P is a hierarchical poset with s elements and t levels and is denoted by $H(s; s_1, s_2, \dots, s_t)$. The **height** of a poset is the maximum length of a chain in the poset. A chain poset is a hierarchical poset over $[s]$ with height s and an antichain poset is a hierarchical poset over $[s]$ with height 1.

Let P and Q be posets defined on disjoint sets X and Y respectively. The **ordinal sum** $P \oplus Q$ of posets P and Q is defined to be the poset on $X \sqcup Y$ such that $i \leq j$ in $P \oplus Q$ if one of the following conditions holds:

(i) $i, j \in P$ and $i \leq j$ in P; (ii) $i, j \in Q$ and $i \leq j$ in Q; (iii) $i \in P$ and $j \in Q$.

Let $\pi : [s] \to \mathbb{N}$ be a map such that $n = \sum_{i=1}^{s} \pi(i)$. The map π is said to be a **labeling** of the poset P, and the pair (P, π) is called a **poset block structure** over $[s]$. Denote $\pi(i)$ by k_i and take V_i as the $\mathbb{F}_q$-vector space $\mathbb{F}_q^{k_i}$ for all $1 \leq i \leq s$. Define V as

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_s$$

which is isomorphic to $\mathbb{F}_q^n$. Each $x \in \mathbb{F}_q^n$ can be written as $x = (x_1, x_2, \ldots, x_s)$ where $x_i \in \mathbb{F}_q^{k_i}, 1 \leq i \leq s$. The **block support** or **π-support** and **(P, π)-weight** of $x \in \mathbb{F}_q^n$ are defined as

$$\mathrm{supp}_\pi(x) = \{i : 1 \leq i \leq s, x_i \neq 0\}$$

and

$$w_{(P,\pi)}(x) = |\langle \mathrm{supp}_\pi(x) \rangle|.$$

For $x, y \in V$,

$$d_{(P,\pi)}(x, y) = w_{(P,\pi)}(x - y)$$

defines a metric over $\mathbb{F}_q^n$ called the **poset block metric** or **(P, π)-metric**. The pair $(\mathbb{F}_q^n, d_{(P,\pi)})$ is said to be a **poset block space**. Also, the pair $(\mathbb{F}_q^n, d_{(\widetilde{P},\pi)})$ is said to be the **dual poset block space** where $\widetilde{P}$ is the dual of the poset P.

A k-dimensional subspace $C \subseteq \mathbb{F}_q^n$ is said to be an (n, k, d) **(P, π)-code**, where $\mathbb{F}_q^n$ is equipped with the poset block metric $d_{(P,\pi)}(\cdot, \cdot)$ and

$$d = d_{(P,\pi)}(C) = \min\{w_{(P,\pi)}(c) : 0 \neq c \in C\}$$

is the (P, π)-minimum distance of C. A generator matrix and a parity check matrix of an (n, k) (P, π)-code are defined as in the classical case and are viewed as blocks with respect to the partition of n corresponding to π. A parity check matrix H is viewed as $H = [H_1\, H_2 \ldots H_s]$ where H_i is an $(n-k) \times k_i$ matrix. The set of blocks $H_{j_1}, H_{j_2}, \ldots, H_{j_t}$ is linearly dependent if the union of all column vectors of $H_{j_1}, H_{j_2}, \ldots, H_{j_t}$ is a linearly dependent set, that is, if there exist $\alpha_i \in V_{j_i}$ such that

$$H_{j_1}\alpha_1^T + H_{j_2}\alpha_2^T + \cdots + H_{j_t}\alpha_t^T = 0$$

and at least one $\alpha_i \neq 0$. Otherwise, the blocks are said to be linearly independent.

The **dual** of an (n, k) code C is defined as

$$C^\perp = \{x \in \mathbb{F}_q^n : x \cdot c = 0 \text{ for all } c \in C\},$$

where $x \cdot c$ is the usual formal inner product on $\mathbb{F}_q^n$. Then $C^\perp$ is an $(n, n-k)$ code. The number of codewords of (P, π)-weight w is denoted by $A_{w,(P,\pi)}(C)$; that is,

$$A_{w,(P,\pi)}(C) = |\{c \in C : w_{(P,\pi)}(c) = w\}|.$$

It is worth noting that when the poset P is taken to be a chain, the poset block metric, introduced by Alves et al. [1], reduces to the NRT block metric [11]. When the label π satisfies $\pi(i) = 1$ for all $i \in [s]$, the poset block metric reduces to poset metric, $d_P(\cdot, \cdot)$, introduced by Brualdi et al. [2]. Similarly, the block metric, $d_\pi(\cdot, \cdot)$, introduced by Feng et al. [4], becomes a particular case of poset block metric when P is taken to be an antichain. When $\pi(i) = 1$ for all $i \in [s]$ and P is an antichain, then the poset block metric reduces to the classical Hamming metric, $d_H(\cdot, \cdot)$.

For $x \in \mathbb{F}_q^n$ and a non-negative integer r, the (P, π)**-ball** with center x and radius r is defined as the set of all those vectors of V whose distance from x relative to poset block metric is less than or equal to r; that is,

$$B_{(P,\pi)}(x, r) = \{y \in \mathbb{F}_q^n : d_{(P,\pi)}(x, y) \leq r\}.$$

A code C is said to be an r**-error correcting** (P, π)-code if the (P, π)-balls of radius r centered at the codewords of C are pairwise disjoint. A code C is said to be an r**-perfect** (P, π)-code if the (P, π)-balls of radius r centered at the codewords of C are pairwise disjoint and their union is $\mathbb{F}_q^n$.

Now we give definition of an additive character. For details on additive characters, refer [7,8]. An **additive character** λ of $\mathbb{F}_q$ is a homomorphism of the additive group $\mathbb{F}_q$ into the multiplicative group of the complex numbers of absolute value 1.

Let f be a complex-valued function defined on $\mathbb{F}_q^n$. The **Fourier transform** $\hat{f}$ of f is defined by

$$\hat{f}(x) = \sum_{y \in \mathbb{F}_q^n} \lambda(x \cdot y) f(y),$$

where λ is a non-trivial additive character of $\mathbb{F}_q$.

Proposition 2.3. (Discrete Poisson summation formula) *Let $C \subseteq \mathbb{F}_q^n$ be a linear code and f be a complex-valued function on $\mathbb{F}_q^n$. Then*

$$\sum_{c' \in C^\perp} f(c') = \frac{1}{|C|} \sum_{c \in C} \hat{f}(c).$$

3. The Singleton bound and MDS (P, π)-codes

In this section, we prove the Singleton bound for the case of poset block metric and introduce the concept of MDS (P, π)-code. In the following theorem, the relation between

(P, π)-weight of a codeword and a parity check matrix of the code in terms of an ideal of the poset P has been derived.

Theorem 3.1. *Let (P, π) be a poset block structure on $\mathbb{F}_q^n$ and $C \subseteq \mathbb{F}_q^n$ be a code with parity check matrix H. Then C has a codeword of (P, π)-weight w if and only if there exists an ideal I of P of cardinality w such that the blocks of H corresponding to I are linearly dependent.*

Proof. Firstly we prove that for any given codeword of C of (P, π)-weight w, there exists an ideal I of P of cardinality w such that the blocks of H corresponding to I are linearly dependent. Consider the parity check matrix $H = [H_1\, H_2 \dots H_s]$ where H_i denotes the i-th block of H. Let $c = (c_1, c_2, \dots, c_s) \in C$ such that $w_{(P,\pi)}(c) = w$. Denote by I the ideal generated by $\mathrm{supp}_\pi(c)$. Since (P, π)-weight of c is w, we get $|I| = |\langle \mathrm{supp}_\pi(c)\rangle| = w$. It is worth noting that

$$c_i = 0 \quad \text{for } i \notin I, \tag{3.1}$$

$$c_i \neq 0 \quad \text{for } i \in M(I). \tag{3.2}$$

Since $c \in C$, therefore $Hc^T = 0$; that is,

$$H_1 c_1^T + H_2 c_2^T + \dots + H_s c_s^T = 0.$$

Using (3.1), we get the required linear dependence relation between the blocks corresponding to I.

Conversely, let there be a linear dependence relation between some blocks of H of the form

$$H_{j_1} \alpha_1^T + H_{j_2} \alpha_2^T + \dots + H_{j_t} \alpha_t^T = 0, \tag{3.3}$$

where $0 \neq \alpha_i \in \mathbb{F}_q^{k_{j_i}}$ for all $1 \leq i \leq t$. We prove that there exists a codeword of (P, π)-weight $w = |\langle j_1, j_2, \dots, j_t\rangle|$. Consider the vector $u = (u_1, u_2, \dots, u_s)$ such that

$$u_{j_i} = \begin{cases} \alpha_i, & \text{for } i = 1, 2, \dots, t, \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

In view of (3.4), (3.3) may be written as

$$H_{j_1} u_{j_1}^T + H_{j_2} u_{j_2}^T + \dots + H_{j_t} u_{j_t}^T = 0$$

which in the expanded form may be written as

$$H_1 u_1^T + H_2 u_2^T + \dots + H_s u_s^T = 0,$$

that is, $Hu^T = 0$. Thus, $u \in C$. In view of (3.4), the codeword u is of weight $w = |\langle j_1, j_2, \ldots, j_t \rangle|$. $\square$

Remark 3.1. It follows from (3.2) that the blocks corresponding to maximal elements of I have non-zero coefficients in the linear dependence relation.

Corollary 3.1. *Let H be a parity check matrix of a (P, π)-code C. The (P, π)-minimum distance of C is d if and only if*

$$d - 1 = \max\{\ell : \text{rank}\{H_{i_1}, H_{i_2}, \ldots, H_{i_\ell}\} = \sum_{j=1}^{\ell} k_{i_j}\} \tag{3.5}$$

for all $\{i_1, i_2, \ldots, i_\ell\} \in \mathcal{I}^\ell(P)$; that is, $d-1$ is the maximum non-negative integer r such that blocks of H corresponding to every ideal in $\mathcal{I}^\ell(P)$ are linearly independent.

In other words, d is the minimum positive integer such that there exists an ideal I of P of cardinality d such that the blocks $\{H_i : i \in I\}$ are linearly dependent.

Remark 3.2. (i) For the case of poset metric, (3.5) reduces to

$$d_P - 1 = \max\{\ell : \text{rank}\{h_{i_1}, h_{i_2}, \ldots, h_{i_\ell}\} = \ell\}$$

for all $\{i_1, i_2, \ldots, i_\ell\} \in \mathcal{I}^\ell(P)$; that is, $d_P - 1$ is the maximum non-negative integer ℓ such that columns of H corresponding to every ideal in $\mathcal{I}^\ell(P)$ are linearly independent. In other words, d_P is the minimum positive integer such that there exists an ideal I of P of cardinality d_P such that the columns $\{h_i : i \in I\}$ are linearly dependent as seen in [2].

(ii) For the case of block metric, (3.5) reduces to

$$d_\pi - 1 = \max\{\ell : \text{rank}\{H_{i_1}, H_{i_2}, \ldots, H_{i_\ell}\} = \sum_{j=1}^{\ell} k_{i_j}\} \tag{3.6}$$

for all $\{i_1, i_2, \ldots, i_\ell\} \subseteq [s]$; that is, $d_\pi - 1$ is the maximum non-negative integer ℓ such that blocks of H corresponding to every ℓ-subset of $[s]$ are linearly independent. In other words, d_π is the minimum positive integer such that there exists a set of d_π linearly dependent blocks of H as seen in [4].

(iii) For the case of NRT block metric, (3.5) reduces to

$$d - 1 = \max\{\ell : \text{rank}\{H_1, H_2, \ldots, H_\ell\} = \sum_{j=1}^{\ell} k_j\};$$

that is, $d-1$ is the maximum positive integer ℓ such that first ℓ blocks of H are linearly independent. In other words, d is the minimum positive integer such that the first d blocks of H are linearly dependent as seen in [11, Theorem 6].

Next we state the Singleton bound for the case of poset block metric and provide two different proofs. The first proof uses Theorem 3.1 whereas the second proof is based on the ideals in the poset analogous to the case of poset metric [5, Proposition 2.1].

Theorem 3.2. *For a (P,π)-code C of length n, dimension k and minimum distance d over $\mathbb{F}_q$, we have*

$$n-k \geq \max_{I \in \mathcal{I}^{d-1}(P)} \{k_{i_1} + k_{i_2} + \cdots + k_{i_{d-1}}\}, \tag{3.7}$$

where the maximum is taken over all ideals of P of cardinality $d-1$.

Proof. Let $H = [H_1 H_2 \ldots H_s]$ be a parity check matrix of the code C. Since the minimum (P,π)-weight of the code is d, it follows from Theorem 3.1 that the blocks of the parity check matrix H corresponding to every ideal $I \in \mathcal{I}^{d-1}(P)$ are linearly independent. Since the maximum number of linearly independent columns of H is $n-k$, we get the result.

Alternative proof. Let $I \in \mathcal{I}^{d-1}(P)$. Consider the map

$$p_{\underset{j \notin I}{\oplus} V_j} : C \to \underset{j \notin I}{\oplus} V_j$$

as the projection of C on the blocks corresponding to $P \setminus I$. Let $p_{\underset{j \notin I}{\oplus} V_j}(x) = p_{\underset{j \notin I}{\oplus} V_j}(y)$. Then $\mathrm{supp}_\pi(x-y) \subseteq I$. This implies that $d_{(P,\pi)}(x,y) \leq d-1$, which is a contradiction since the minimum distance of C is d. Therefore, $p_{\underset{j \notin I}{\oplus} V_j}$ is injective. Hence

$$|C| \leq |\underset{j \notin I}{\oplus} V_j|.$$

Therefore, $\dim C \leq n - (k_{i_1} + k_{i_2} + \cdots + k_{i_{d-1}})$. Thus

$$n-k \geq k_{i_1} + k_{i_2} + \cdots + k_{i_{d-1}}.$$

Since this is true for any ideal $I \in \mathcal{I}^{d-1}(P)$, we get (3.7). $\square$

Remark 3.3. (i) For an (n,k,d_π) linear error block code C over $\mathbb{F}_q$ with the partition π given by $n = k_1 + k_2 + \cdots + k_s$, $k_1 \geq k_2 \geq \cdots \geq k_s$, (3.7) reduces to

$$n-k \geq k_1 + k_2 + \cdots + k_{d_\pi - 1},$$

which is same as the Singleton bound for the case of block metric [4, Theorem 2.1].
(ii) For an (n,k,d_P) poset code C over $\mathbb{F}_q$, (3.7) reduces to

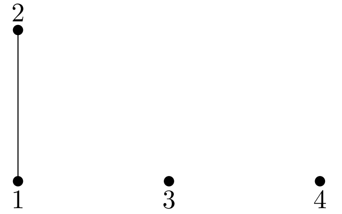
$$d_P \leq n - k + 1,$$

which is same as the Singleton bound for the case of poset metric [5, Corollary 2.2].

We now define a maximum distance separable (P, π)-code.

Definition 3.1. Let (P, π) be a poset block structure on $\mathbb{F}_q^n$. A linear code $C \subseteq \mathbb{F}_q^n$ is called a maximum distance separable (MDS)(P, π)-code if it attains the Singleton bound.

Example 3.1. Consider the poset P given by the following Hasse diagram:



Let $k_1 = 1, k_2 = 3, k_3 = 2$, and $k_4 = 1$. Consider the (P, π)-code C generated by the following generator matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Then C is a $(7, 3)$ code with $d = d_{(P,\pi)}(C) = 3$. We have

$$\mathcal{I}^2(P) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{3, 4\}\}.$$

Thus

$$\max_{I \in \mathcal{I}^2(P)} \{k_{i_1} + k_{i_2}\} = 4 = n - k.$$

Therefore, C is an MDS (P, π)-code.

4. *I*-perfect codes

In this section, we extend the concept of I-balls originally defined for poset metric (see [5]) to the case of poset block metric and study their properties. For the sake of simplicity, we retain the same notation as used by Hyun and Kim [5].

Definition 4.1. Let (P, π) be a poset block structure on $\mathbb{F}_q^n$ and $I \in \mathcal{I}(P)$. For $x \in \mathbb{F}_q^n$, the I-ball centered at x is defined as

$$B_I(x) = \{y \in \mathbb{F}_q^n : \langle \mathrm{supp}_\pi(x-y) \rangle \subseteq I\}.$$

We denote the I-ball centered at 0 by B_I. Wherever required, we use the notation $B_{I,(P,\pi)}(x)$ to specify the metric under consideration.

Remark 4.1. (i) Since $\mathrm{supp}_\pi(x-y) \subseteq I$ if and only if $\langle \mathrm{supp}_\pi(x-y) \rangle \subseteq I$, we have

$$B_I(x) = \{y \in \mathbb{F}_q^n : \mathrm{supp}_\pi(x-y) \subseteq I\}.$$

(ii) Two vectors x and y belong to the same I-ball for some $I \in \mathcal{I}^r(P)$ if and only if $d_{(P,\pi)}(x,y) \leq r$.
(iii) The (P,π)-ball of radius r centered at x is union of I-balls centered at x where $I \in \mathcal{I}^r(P)$; that is,

$$B_{(P,\pi)}(x,r) = \bigcup_{I \in \mathcal{I}^r(P)} B_I(x).$$

Next, we state the basic facts about I-balls in poset block metric. The following proposition is a generalization of [5, Proposition 2.7] wherein the metric considered was poset metric. The proof is on similar lines and hence omitted.

Proposition 4.1. *Let (P,π) be a poset block structure on $\mathbb{F}_q^n$ and $I \in \mathcal{I}(P)$. Then*

1. *B_I is a subspace of $\mathbb{F}_q^n$, namely $\underset{i\in I}{\oplus} V_i$, of dimension $\sum\limits_{i\in I} k_i$.*
2. *For $x \in \mathbb{F}_q^n$, $B_I(x)$ is a coset of B_I containing x; that is, $B_I(x) = x + B_I$.*
3. *For $x, y \in \mathbb{F}_q^n$, $B_I(x)$ and $B_I(y)$ are either disjoint or identical.*
4. *The full space $\mathbb{F}_q^n$ is partitioned into I-balls.*

We now define an I-perfect code for the case of poset block metric.

Definition 4.2. Let (P,π) be a poset block structure on $\mathbb{F}_q^n$ and $I \in \mathcal{I}(P)$. A linear (P,π)-code $C \subseteq \mathbb{F}_q^n$ is said to be I-perfect if the I-balls centered at the codewords of C are pairwise disjoint and their union is $\mathbb{F}_q^n$; that is,

$$\mathbb{F}_q^n = \bigsqcup_{x\in C} B_I(x).$$

The following lemma is a consequence of the definition of an I-perfect code.

Lemma 4.1. *The following are equivalent:*

1. *C is an I-perfect code.*

2. (i) *(The covering condition)* $\sum_{i \in I} k_i = n - k$.
   (ii) *(The packing condition)* $|C \cap B_I| = 1$.
3. $|C \cap B_I(x)| = 1$ *for all* $x \in \mathbb{F}_q^n$*; that is, each element of* $\mathbb{F}_q^n$ *belongs to precisely one* I*-ball centered at a codeword of* C*.*

The following proposition shows that I-perfect codes are systematical with information symbols in blocks corresponding to $P \setminus I$ and parity check symbols in blocks corresponding to I. This shows that it is easier to deal with I-perfect codes than r-perfect codes. In what follows, we write $x \in \mathbb{F}_q^n$ as (x_1, x_2) where $x_1 \in \bigoplus_{j \notin I} V_j$ and $x_2 \in \bigoplus_{i \in I} V_i$.

Proposition 4.2. *An* (n, k) (P, π)*-code is* I*-perfect if and only if there is a function*

$$f : \bigoplus_{j \notin I} V_j \to \bigoplus_{i \in I} V_i \tag{4.1}$$

such that

$$C = \{(v, f(v)) : v \in \bigoplus_{j \notin I} V_j\}. \tag{4.2}$$

Proof. Assume that the code C is I-perfect. Let $v \in \bigoplus_{j \notin I} V_j$. Since C is an I-perfect code, there exists $c \in C$ such that $(v, 0) \in B_I(c)$ which implies that $c - (v, 0) = (0, u) \in B_I$. This proves that $c = (v, u)$. Moreover, if there is another element $c' = (v, w) \in C$, then $c - c' = (0, u - w) \in B_I$ which implies that $c \in B_I(c')$. This contradicts that C is an I-perfect code. Therefore, we can define a function $f : \bigoplus_{j \notin I} V_j \to \bigoplus_{i \in I} V_i$ which sends $v \in \bigoplus_{j \notin I} V_j$ to the unique $u \in \bigoplus_{i \in I} V_i$ such that $c = (v, u)$, and clearly, every element of C is of the form $(v, f(v))$.

Conversely, assume that such a function exists. Then, for any $(v, u) \in \mathbb{F}_q^n$, we have $C \cap B_I(v, u) = \{(v, f(v))\}$. By Lemma 4.1(3), we get that C is an I-perfect code. $\square$

Note that we do not use the linearity of C anywhere in the above proof. Hence, the above proposition holds for non-linear codes as well. However, the following remark holds just in the case of linear codes.

Remark 4.2. Such a function is a linear transformation since C is a linear subspace of $\mathbb{F}_q^n$.

Proof. First, $f(0) = 0$ since $(0, 0) \in C$. Let $v, w \in \bigoplus_{j \notin I} V_j$. Then $(v, f(v)), (w, f(w)) \in C$. This implies that $\alpha(v, f(v)) + \beta(w, f(w)) \in C$ for $\alpha, \beta \in \mathbb{F}_q$, which gives us $(\alpha v + \beta w, \alpha f(v) + \beta f(w)) \in C$ and hence, $f(\alpha v + \beta w) = \alpha f(v) + \beta f(w)$. Thus, f is a linear transformation. $\square$

We now explore the connection between r-perfectness and I-perfectness of a (P,π)-code. We generalize [6, Proposition 3.1] to derive a necessary and sufficient condition for a poset block code C to be an r-error correcting code in terms of I-balls.

Theorem 4.1. *A (P,π)-code C is an r-error correcting code if and only if*

$$c - c' \notin B_{I' \cup I''} \text{ for each } I', I'' \in \mathcal{I}^r(P) \text{ and } c, c' \in C, c \neq c'. \tag{4.3}$$

Proof. Assume that C is an r-error correcting code. Suppose there exist $c, c' \in C, c \neq c'$ and $I', I'' \in \mathcal{I}^r(P)$ such that $c - c' \in B_{I' \cup I''}$. Consider $u = c - p_{B_{I'}}(c - c')$, where $p_{B_{I'}}$ denotes the projection of C on blocks corresponding to I'. Then

$$d_{(P,\pi)}(u, c) = w_{(P,\pi)}(p_{B_{I'}}(c - c')) \leq |I'| = r.$$

Also, we have that

$$\begin{aligned} d_{(P,\pi)}(u, c') &= w_{(P,\pi)}((c - c') - p_{B_{I'}}(c - c')) \\ &= w_{(P,\pi)}(p_{B_{I''}}(c - c')) \\ &\leq |I''| = r. \end{aligned}$$

This implies that $u \in B_{(P,\pi)}(c, r) \cap B_{(P,\pi)}(c', r)$, which is a contradiction.

Conversely, assume that C is not an r-error correcting code. Then there exists $c, c' \in C, c \neq c'$ and $x \in \mathbb{F}_q^n$ such that

$$x \in B_{(P,\pi)}(c, r) \cap B_{(P,\pi)}(c', r)$$

This implies that $d_{(P,\pi)}(x, c) \leq r$ and $d_{(P,\pi)}(x, c') \leq r$. We have $|\langle \mathrm{supp}_\pi(c - x)\rangle| \leq r$ and $|\langle \mathrm{supp}_\pi(c' - x)\rangle| \leq r$. Using Remark 4.1(ii), we get $\mathrm{supp}_\pi(c - x) \subseteq I'$ and $\mathrm{supp}_\pi(c' - x) \subseteq I''$ for some $I', I'' \in \mathcal{I}^r(P)$. This, in turn, implies that $\mathrm{supp}_\pi(c - c') \subseteq I' \cup I''$ which gives $c - c' \in B_{I' \cup I''}$, which contradicts (4.3). This proves the result. □

Remark 4.3. In particular for $I' = I''$, the above theorem implies that for an r-error correcting poset block code, the I-balls centered at the codewords of C are disjoint for each $I \in \mathcal{I}^r(P)$. This fact also follows from Remark 4.1(iii).

Next, we prove a lemma which establishes equivalence between the r-perfectness and I-perfectness of a (P,π)-code when P has a unique ideal of cardinality r.

Lemma 4.2. *Let (P,π) be a poset block structure with $\mathcal{I}^r(P) = \{I\}$. Then a code C is an r-perfect (P,π)-code if and only if C is an I-perfect (P,π)-code.*

Proof. Since C is r-perfect (P, π)-code, we have

$$\mathbb{F}_q^n = \bigsqcup_{c \in C} B_{(P,\pi)}(c, r).$$

Since $\mathcal{I}^r(P) = \{I\}$ and using Remark 4.1(iii), the above is equivalent to

$$\mathbb{F}_q^n = \bigsqcup_{c \in C} B_I(c, r)$$

which proves that C is I-perfect. □

Remark 4.4. In the case of the NRT block metric, $|\mathcal{I}^r(P)| = 1$ for each $1 \leq r \leq s$. Therefore, C is an r-perfect NRT block code if and only if C is an I-perfect NRT block code for $I \in \mathcal{I}^r(P)$. Using Proposition 4.2, every r-perfect NRT block code is systematical as seen in [1, Proposition 3.1]. In particular when all the block dimensions are 1, that is, for the NRT metric, no r-perfect codes exist for $r < n - k$.

Using Lemma 4.2 for $r = n - k$, a complete characterization of an $(n - k)$-perfect (P, π)-code can be obtained as follows.

Theorem 4.2. *An (n, k) (P, π)-code C is a $(n - k)$-perfect code if and only if $\mathcal{I}^{n-k}(P) = \{I\}$ and C is an I-perfect code.*

Proof. Let C be an $(n - k)$-perfect code. If $\{I\} \subsetneq \mathcal{I}^{n-k}(P)$, then

$$|B_{(P,\pi)}(0, n - k)| > |B_I(0)| = q^{n-k}$$

which is a contradiction. Hence, $\mathcal{I}^{n-k}(P) = \{I\}$. Using Lemma 4.2, C is an I-perfect code. The converse also follows from Lemma 4.2. □

Remark 4.5. In this theorem, $\mathcal{I}^{n-k}(P) = \{I\}$ implies that the $(n-k)$ blocks in I form a chain, say $1 \leq 2 \leq \cdots \leq n - k$. Also, since C is I-perfect, from Lemma 4.1 we have that $k_i = 1$ for all $i \in I$. Thus, Theorem 4.2 can be restated as follows:

An (n, k) (P, π)-code C is an $(n-k)$-perfect code if and only if $\Gamma^i(P) = \{i\}$ and $k_i = 1$ for $1 \leq i \leq n - k$.

In [5, Theorem 2.12], Hyun and Kim have derived a relation between MDS poset codes and I-perfect codes. Generalizing the same in the following theorem, we prove that every MDS poset block code is an I-perfect code for some ideal of P of cardinality $d - 1$ and vice-versa.

Theorem 4.3. *Let (P, π) be a poset block structure on $\mathbb{F}_q^n$ and $C \subseteq \mathbb{F}_q^n$ be an (n, k, d) (P, π)-code. Then C is an MDS (P, π)-code if and only if C is an I-perfect code for some $I \in \mathcal{I}^{d-1}(P)$.*

Proof. Let C be an MDS (P,π)-code. Then

$$n-k=\max_{I\in\mathcal{I}^{d-1}(P)}\{k_{i_1}+k_{i_2}+\cdots+k_{i_{d-1}}\}.$$

Let $I\in\mathcal{I}^{d-1}(P)$ be such that $\sum\limits_{i\in I}k_i=n-k$. Suppose the I-balls centered at c and c' intersect. Since two I-balls are either distinct or identical, therefore we have $c\in B_I(c')$ for some $c,c'\in C, c\neq c'$. Then $\mathrm{supp}_\pi(c-c')\subseteq I$. This gives $|\mathrm{supp}_\pi(c-c')|\leq d-1$. This implies that $w_{(P,\pi)}(c-c')\leq d-1$, which is a contradiction. It follows that the I-balls centered at the codewords of C are disjoint and hence, cover $\mathbb{F}_q^n$. Therefore, C is an I-perfect code. Conversely, let C be an I-perfect code for some $I\in\mathcal{I}^{d-1}(P)$. Then $\sum\limits_{i\in I}k_i=n-k$. Thus,

$$\max_{I\in\mathcal{I}^{d-1}(P)}\{k_{i_1}+k_{i_2}+\cdots+k_{i_{d-1}}\}=n-k.$$

Hence, C is an MDS (P,π)-code. $\square$

Remark 4.6. In Example 3.1, C is I-perfect for $\{1,2\}\in\mathcal{I}^2(P)$. In general, the ideal may not be unique. In fact, an (n,k,d) MDS (P,π)-code is I-perfect for all $I\in\mathcal{I}^{d-1}(P)$ such that $\sum\limits_{i\in I}k_i=n-k$.

The next result shows that the dual of an I-perfect (P,π)-code is an I^c-perfect $(\widetilde{P},\pi)$-code. Firstly, we state a lemma which is a generalization of [5, Lemma 3.11]. The proof is on similar lines and therefore we omit it. We denote by $\chi(E,\cdot)$ the indicator function of a subset E of $\mathbb{F}_q^n$.

Lemma 4.3. *Let (P,π) be a poset block structure with $I\in\mathcal{I}(P)$. For $x,y\in\mathbb{F}_q^n$, we have*

$$\hat{\chi}(B_{I,(P,\pi)}(y)|x)=\lambda(x\cdot y)q^{\sum\limits_{i\in I}k_i}\chi(B_{I^c,(\widetilde{P},\pi)}|x).$$

In particular,

$$\hat{\chi}(B_{I,(P,\pi)}|x)=q^{\sum\limits_{i\in I}k_i}\chi(B_{I^c,(\widetilde{P},\pi)}|x).$$

Theorem 4.4. *Let (P,π) be a poset block structure and $I\in\mathcal{I}(P)$. A code C is an I-perfect (P,π)-code if and only if $C^\perp$ is an I^c-perfect $(\widetilde{P},\pi)$-code where $\widetilde{P}$ is the dual of P.*

Proof. Let C be an I-perfect (P,π)-code. By Proposition 2.2, $I^c\in\mathcal{I}(\widetilde{P})$ with $\sum\limits_{i\in I^c}k_i=k$ as $\sum\limits_{i\in I}k_i=n-k$. Using Proposition 2.3, we have

$$|C^{\perp} \cap B_{I^c,(\widetilde{P},\pi)}(x)| = \sum_{c' \in C^{\perp}} \chi(B_{I^c,(\widetilde{P},\pi)}(x)|c')$$
$$= \frac{1}{|C|} \sum_{c \in C} \hat{\chi}(B_{I^c,(\widetilde{P},\pi)}(x)|c)$$

Applying Lemma 4.3, we get

$$|C^{\perp} \cap B_{I^c,(\widetilde{P},\pi)}(x)| = \frac{1}{|C|} \sum_{c \in C} \lambda(c \cdot x) q^{\sum\limits_{i \in I^c} k_i} \chi(B_{I,(P,\pi)}|c)$$
$$= \sum_{c \in C \cap B_{I,(P,\pi)}} \lambda(c \cdot x).$$

Since C is an I-perfect code, we get $C \cap B_{I,(P,\pi)} = \{0\}$. Therefore

$$|C^{\perp} \cap B_{I^c,(\widetilde{P},\pi)}(x)| = \lambda(0) = 1.$$

Using Lemma 4.1, $C^{\perp}$ is an I^c-perfect $(\widetilde{P}, \pi)$-code. The converse is just the dual statement. □

5. Duality and weight distribution

In this section, we explore the case for an MDS (P, π)-code when all the blocks have the same dimension. The results presented in this section are generalizations of the results for the case of poset metric given in [5] which would reduce to the corresponding results for poset metric for $m = 1$. When all the blocks have the same dimension, there is a direct relation between n, k and d, as in the classical case. Let (P, π) be a poset block structure such that $k_1 = k_2 = \cdots = k_s = m$. Then the Singleton bound becomes

$$d \leq \frac{n-k}{m} + 1. \tag{5.1}$$

The following theorem gives characterizations of an MDS (P, π)-code when all the blocks have the same dimension. In this case, we prove that the duality theorem holds for an MDS (P, π)-code. Further, a code is MDS (P, π)-code if and only if there is a uniform distribution of its codewords among the I-balls.

Theorem 5.1. *Let P be a poset on $[s]$ and $\widetilde{P}$ be its dual poset. Let π be a labeling of the poset P with $k_1 = k_2 = \cdots = k_s = m$. Let C be an (n, k) (P, π)-code. Then the following statements are equivalent:*

1. *C is an MDS (P, π)-code;*
2. *C is an I-perfect (P, π)-code for all $I \in \mathcal{I}^{\frac{n-k}{m}}(P)$;*
3. *$C^{\perp}$ is an I-perfect $(\widetilde{P}, \pi)$-code for all $I \in \mathcal{I}^{\frac{k}{m}}(\widetilde{P})$;*

4. *$C^\perp$ is an MDS $(\widetilde{P}, \pi)$-code;*
5. *For any ideal I of P and $x \in \mathbb{F}_q^n$,*

$$|B_I(x) \cap C| = \begin{cases} q^{m|I|-n+k} & \text{if } |I| \geq \frac{n-k}{m}, \\ 1 & \text{if } |I| < \frac{n-k}{m} \text{ and } x \in \bigcup_{c \in C} B_I(c), \\ 0 & \text{if } |I| < \frac{n-k}{m} \text{ and } x \notin \bigcup_{c \in C} B_I(c). \end{cases} \tag{5.2}$$

Proof. The first four equivalences follow immediately from Theorem 4.3 and Theorem 4.4. We now prove the equivalence of (1) and (5). Firstly we assume that C is an MDS (P, π)-code. Let $I \in \mathcal{I}^r(P)$ and $x \in \mathbb{F}_q^n$. There are two cases.
Case I. For $r < \frac{n-k}{m}$, using Lemma 4.1, no two codewords of C belong to the same I-ball. Also, since two I-balls are either disjoint or identical, it follows that

$$|B_I(x) \cap C| = \begin{cases} 1 & \text{if } x \in \bigcup_{c \in C} B_I(c), \\ 0 & \text{if } x \notin \bigcup_{c \in C} B_I(c). \end{cases}$$

Case II. For $r \geq \frac{n-k}{m}$, by Proposition 2.1, there exists an ideal $J \in I^{\frac{n-k}{m}}(P)$ such that $J \subseteq I$. Since B_J is a subspace of B_I of codimension $(|I| - |J|)m$, therefore the number of cosets of B_J in B_I is $q^{(|I|-|J|)m}$. This also holds for $B_I(x)$ for any $x \in \mathbb{F}_q^n$. Therefore

$$|B_I(x) \cap C| = q^{(|I|-|J|)m} |B_J(x) \cap C|$$

Since C is J-perfect, using Theorem 4.3, we get

$$\begin{aligned} |B_I(x) \cap C| &= q^{(|I|-|J|)m} \\ &= q^{m|I|-n+k}. \end{aligned}$$

Conversely, assume that for any ideal I of P and $x \in \mathbb{F}_q^n$, we have (5.2). In particular, for any $I \in I^{\frac{n-k}{m}}(P)$ and $x \in \mathbb{F}_q^n$, we get $|B_I(x) \cap C| = q^0 = 1$ which proves that C is an I-perfect code for all $I \in \mathcal{I}^{\frac{n-k}{m}}(\widetilde{P})$ and hence an MDS (P, π)-code. □

Remark 5.1. For the case of block metric, Theorem 5.1 reduces to [4, Theorem 4.1]. In addition, we have that a code is an MDS block code if and only if there are q^δ codewords of C in every set of $(\frac{n-k}{m} + \delta)$ blocks.

Next we determine the weight distribution of an MDS (P, π)-code for the case when all the blocks have the same dimension.

Theorem 5.2. *Let (P, π) be a poset block structure with $k_1 = k_2 = \cdots = k_s = m$ and C be a (n, k, d) MDS (P, π)-code over $\mathbb{F}_q$. Then*

$$A_{w,(P,\pi)}(C) = \begin{cases} 1 & \textit{if } w = 0, \\ 0 & \textit{if } 1 \leq w \leq d-1 \\ (q^m - 1) \sum\limits_{I \in \mathcal{I}^w(P)} \sum\limits_{j=0}^{w-d} (-1)^j \binom{|M(I)-1}{j} q^{(w-d-j)m} & \textit{if } w \geq d. \end{cases}$$

Proof. The proof is on similar lines as in the case of poset metric [5, Theorem 3.4] and hence omitted. □

For the case of RT-metric, we can obtain the number of codewords of (P, π)-weight w since we can count the number of ideals of P of cardinality w with i maximal elements [14] as follows.

Corollary 5.1. *Let P be a disjoint union of p chains of size t each and π be a labeling of the poset P with $k_1 = k_2 = \cdots = k_s = m$ and C be an (n, k, d) MDS (P, π)-code over $\mathbb{F}_q$. Then*

$$A_{w,(P,\pi)}(C) = \begin{cases} 1 & \textit{if } w = 0, \\ 0 & \textit{if } 1 \leq w \leq d-1, \\ (q^m - 1) \sum\limits_{l=1}^{p} \binom{p}{l} \sigma_t(l, w) \sum\limits_{j=0}^{w-d} (-1)^j \binom{l-1}{j} q^{(w-d-j)m} & \textit{if } w \geq d \end{cases}$$

where

$$\sigma_t(l, w) = |\{(a_1, a_2, \ldots, a_l) \in \mathbb{N}^l : a_1 + a_2 + \cdots + a_l = w, 0 \leq a_i \leq t, 1 \leq l \leq p\}|.$$

Proof. The result follows from the fact that the number of ideals of P of cardinality w with l maximal elements is given by $\binom{p}{l}\sigma_t(l, w)$ (see [14, Lemma 3.2]). □

Remark 5.2. (i) In the above corollary by taking $t = 1$, P becomes an antichain of cardinality s and the metric reduces to block metric. In this case, for an (n, k, d_π) MDS block code C, we have

$$A_{w,\pi}(C) = (q^m - 1) \binom{s}{w} \sum_{j=0}^{w-d_\pi} (-1)^j \binom{w-1}{j} q^{(w-d_\pi-j)m} \text{ for } w \geq d_\pi$$

as seen in [4, Theorem 4.4].

(ii) In Corollary 5.1, taking $p = 1$, P becomes a chain of cardinality s and the metric reduces to NRT block metric. In this case, for an (n, k, d) MDS NRT block code C, we have

$$A_{w,(P,\pi)}(C) = (q^m - 1) q^{(w-d)m} \text{ for } w \geq d.$$

For a hierarchical poset block metric, Theorem 5.2 yields the following result.

Corollary 5.2. *Let $P = H(s; s_1, s_2, \ldots, s_t)$ be a hierarchical poset of s elements with t-levels and C be an (n, k, d) MDS (P, π)-code over $\mathbb{F}_q$. Then, for $w \geq d$, we have*

$$A_{w,(P,\pi)}(C) = (q^m - 1)\binom{s_l}{w - (s_1 + s_2 + \cdots + s_{l-1})}$$
$$\times \sum_{j=0}^{w-d} (-1)^j \binom{w - (s_1 + s_2 + \cdots + s_{l-1}) - 1}{j} q^{(w-d-j)m}$$

where $l = l(w)$ is defined to be the integer satisfying

$$(s_1 + s_2 + \cdots + s_{l-1}) < w \leq (s_1 + s_2 + \cdots + s_l).$$

In the following proposition, we generalize [5, Proposition 3.13] for an (n, k, d) MDS (P, π)-code.

Proposition 5.1. *Let (P, π) be a poset block structure with $k_1 = k_2 = \cdots = k_s = m$ and C be an (n, k, d) MDS (P, π)-code over $\mathbb{F}_q$. Then we have the following inequalities:*

$$q^m \geq \frac{n-k}{m} + 1 - \frac{\sum\limits_{I \in \mathcal{I}^{\frac{n-k}{m}+2}(P)} |I_M|}{|\mathcal{I}^{\frac{n-k}{m}+2}(P)|} \quad \text{if } \frac{k}{m} \geq 2, \tag{5.3}$$

$$q^m \geq \frac{k}{m} + 1 - \frac{\sum\limits_{I \in \mathcal{I}^{\frac{n-k}{m}+2}(P)} |I_M^c|}{|\mathcal{I}^{\frac{n-k}{m}+2}(P)|} \quad \text{if } \frac{k}{m} \leq \frac{n}{m} - 2. \tag{5.4}$$

Proof. Using Theorem 5.2 for $r = \frac{n-k}{m} + 2$, we have

$$A_{\frac{n-k}{m}+2,(P,\pi)(C)} = (q^m - 1) \sum_{I \in \mathcal{I}^{\frac{n-k}{m}}(P)} (q^m - \left(\frac{n-k}{m} + 1 - |I_M|\right))$$
$$= (q^m - 1)[|\mathcal{I}^{\frac{n-k}{m}}(P)|(q^m - \left(\frac{n-k}{m} + 1\right)) + \sum_{I \in \mathcal{I}^{\frac{n-k}{m}}(P)} |I_M|].$$

Since $A_{\frac{n-k}{m}+2,(P,\pi)(C)} \geq 0\,(\frac{k}{m} \geq 2)$, we get (5.3). By Theorem 5.1, $C^\perp$ is an $(n, n-k)$ MDS $(\widetilde{P}, \pi)$-code. Applying the above argument to $C^\perp$, we get (5.4). □

Remark 5.3. For the case of block metric, the above proposition reduces to [4, Theorem 4.4].

In what follows, we show that the class of MDS poset block codes is quite large. The following proposition is a generalization of [5, Proposition 4.4] and proves that every

$(\frac{n-k}{m})$-perfect (P,π)-code is an MDS (P,π)-code when all the blocks have the same dimension.

Proposition 5.2. *Let (P,π) be a poset block structure with $k_1 = k_2 = \cdots = k_s = m$ and C be an (n,k) code over $\mathbb{F}_q$. If C is $(\frac{n-k}{m})$-perfect (P,π)-code then C is an MDS (P,π)-code.*

Proof. Let $I \in \mathcal{I}^{\frac{n-k}{m}}(P)$. Suppose $c \in B_I(c')$ where $0 \neq c, c' \in C$ and $c \neq c'$. Then $\langle \mathrm{supp}_\pi(c - c')\rangle \subseteq I$. Since C is an $(\frac{n-k}{m})$-perfect (P,π)-code, therefore the minimum weight of C is at least $\frac{n-k}{m} + 1$. Thus, we get

$$\frac{n-k}{m} + 1 \leq w_{(P,\pi)}(c - c') \leq |I| = \frac{n-k}{m}$$

which is a contradiction. Hence, C is I-perfect for all $I \in \mathcal{I}^{\frac{n-k}{m}}(P)$. Using Theorem 5.1, C is an MDS (P,π)-code. $\square$

The following example shows that the converse of the above proposition does not hold.

Example 5.1. Consider P to be an antichain on the set $\{1,2\}$. Consider $\mathbb{F}_3^4$ with the (P,π)-metric where $k_1 = k_2 = 2$. Let C be the $(4,2)$ code generated by the following generator matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

For the code C, $d = d_{(P,\pi)}(C) = 2$. Since

$$d = \frac{4-2}{2} + 1 = 2,$$

therefore C is an MDS (P,π)-code. Denote $(10,12)$ and $(21,02)$ by c and c', respectively. Then

$$(10,02) \in B_{(P,\pi)}(c,1) \cap B_{(P,\pi)}(c',1).$$

This implies that C is not a 1-perfect code.

Remark 5.4. It follows from Theorem 5.1 and Lemma 4.2 that the converse of Proposition 5.2 holds when $|\mathcal{I}^{\frac{n-k}{m}}(P)| = 1$.

Lemma 5.1. *Let P, Q be posets on $[s]$ such that Q is finer than P. Let π be a labeling of the poset P with $k_1 = k_2 = \cdots = k_s = m$. If C is a MDS (P,π)-code then C is an MDS (Q,π)-code.*

Proof. Since Q is finer than P and the labeling π is fixed, we have $d_{(P,\pi)}(x,y) \leq d_{(Q,\pi)}(x,y)$ which implies that $d_{(P,\pi)}(C) \leq d_{(Q,\pi)}(C)$. Using (5.1),

$$\frac{n-k}{m} + 1 = d_{(P,\pi)}(C) \leq d_{(Q,\pi)}(C) \leq \frac{n-k}{m} + 1,$$

which implies that $d_{(Q,\pi)}(C) = \frac{n-k}{m} + 1$. Hence C is an MDS (Q,π)-code. $\square$

Corollary 5.3. *An MDS block code C is an also an MDS poset block code for every poset defined on the set of blocks.*

Example 5.2. Let $\pi(1) = \pi(2) = \pi(3) = 3$. Consider the code C generated by the following generator matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Then C is an $(9,3)$ code with $d_\pi = 3$. Therefore

$$\frac{n-k}{m} + 1 = d_\pi.$$

Hence, C is an MDS block code. Using Corollary 5.3, C is an MDS (P,π)-code for every poset P defined on $\{1,2,3\}$.

Theorem 5.3. *For every (n,k) code C over $\mathbb{F}_q$ and a positive integer m such that m divides n and k, there exists a (P,π)-metric with $k_1 = k_2 = \cdots = k_s = m$ for which C is an MDS (P,π)-code.*

Proof. Consider an information set of C, say A. By definition, $A \subseteq [n]$ with $|A| = k$ and $c \in C$ is non-zero if and only if its projection on A is non-zero. Since m divides k, we divide A into blocks of dimension m each. Also, m divides $n-k$, thus we divide the remaining positions, namely $[n] \setminus A$, into blocks of dimension m each. Let P_1 and P_2 be arbitrary posets on the blocks corresponding to A and $[n] \setminus A$ respectively. We denote by P the ordinal sum of P_1 and P_2. If $c \in C$ is non-zero, then there exists $i \in A$ such that $c_i \neq 0$. Therefore,

$$w_{(P,\pi)}(c) \geq w_{(P,\pi)}(c_i) = \frac{n-k}{m} + 1.$$

Since $c \neq 0$ was an arbitrary codeword, we have $d \geq \frac{n-k}{m} + 1$. It follows from (5.1) that C is an MDS (P,π)-code. $\square$

Remark 5.5. In particular, we have that for any code C, there always exists a poset block metric for which C is an MDS poset block code.

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