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The packing radius of a poset block code

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We investigate the properties of the packing radius of a code with respect to poset block metric. In the process, we have addressed a few minor errors in the paper, "The packing radius of a code and partitioning problems: The case for poset metrics", in *Proc. IEEE Int. Symp. Information Theory* (2014), pp. 2954–2958 by D'Oliveira and Firer.

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1. Introduction

The packing radius of a code with respect to a metric d, denoted as R_d, is an important concept of coding theory. It is well known [8, Theorem 1.11.4] that nearest neighbor decoding decodes a received vector with R_d or fewer errors correctly but will not always decode correctly a received vector with $R_d + 1$ errors. The packing radius of a code can be obtained directly from its minimum distance in the case of the classical Hamming metric. However, for the poset metric, it is just known that [4]:

$$\left\lfloor \frac{d_{P_n}(C) - 1}{2} \right\rfloor \leq R_{d_{P_n}}(C) \leq d_{P_n}(C) - 1. \tag{1.1}$$

It is remarked in [4] that the lower bound is attained for any linear code if $d_{P_n} = d_H$, i.e., when the poset is an anti-chain, and the upper bound is attained for any linear

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code if the poset is a chain. We show that there are posets which are not anti-chains or chains for which the bounds of (1.1) are attained. The counter examples of such posets are given in Sec. 3.

To the best of our knowledge, the packing radius of a poset block code was only known for an NRT block code [10], i.e., when the poset is a chain, and a linear error block code [3], i.e., when the poset is an anti-chain. In general, the packing radius of a code with respect to poset block metric [1] follows the inequality:

$$\left\lfloor \frac{d_{(P_s,\pi)}(C)-1}{2} \right\rfloor \leq R_{d_{(P_s,\pi)}}(C) \leq d_{(P_s,\pi)}(C)-1. \tag{1.2}$$

The lower bound is attained when P_s is an anti-chain and the upper bound is attained when P_s is a chain.

In this paper, we show that analogous to the case of poset metric, the determination of packing radius of a single vector in poset block metric is equivalent to an NP-hard problem viz. "the poset partition problem". The method developed in the case of poset metric to compare the packing radii of two vectors without calculating them explicitly is now applicable in the context of poset block metric. We derive the packing radius of a code with respect to poset block metric when the poset is hierarchical and to some extent when the poset is a disjoint union of chains of equal size. The relationship between the packing radius of a poset block code and that of the corresponding poset and of block codes has been discussed.

2. Preliminaries

In this section, we discuss basic definitions and results.

Let $P_s = ([s], \leq)$ denote a **poset** with s elements $[s] = \{1, 2, \ldots, s\}$ where $\leq$ is a partial order on $[s]$. An **ideal** I of the poset P_s is a subset of $[s]$ such that $x \in I$, $y \leq x$ implies $y \in I$. For $X \subseteq [s]$, we denote by $\langle X \rangle$ the smallest ideal containing X, called the **ideal generated by** X. We denote the ideal generated by $X = \{i\}$ as $\langle i \rangle$ (rather than $\langle \{i\} \rangle$).

Let $\pi : [s] \to \mathbb{N}$ be a map such that $n = \sum_{i=1}^{s} \pi(i)$. The map π is said to be a **labeling** of the poset P_s and the pair (P_s, π) is called a **poset block structure** over $[s]$. Denote $\pi(i)$ by k_i and take V_i as the $\mathbb{F}_q$-vector space $V_i = \mathbb{F}_q^{k_i}$ for every $1 \leq i \leq s$. Define V as

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_s$$

which is isomorphic to $\mathbb{F}_q^n$. Each $x \in \mathbb{F}_q^n$ can be uniquely decomposed as $x = (x_1 x_2 \ldots x_s)$ where $x_i \in \mathbb{F}_q^{k_i}, 1 \leq i \leq s$. The **block support** or **π-support** and **(P_s, π)-weight** of $x \in \mathbb{F}_q^n$ are defined, respectively, as

$$\mathrm{supp}_\pi(x) = \{i : 1 \leq i \leq s, x_i \neq 0\}$$

and

$$w_{(P_s,\pi)}(x) = |\langle \mathrm{supp}_\pi(x) \rangle|$$

where $|\cdot|$ denotes the cardinality of the given set. For $x, y \in V$,

$$d_{(P_s,\pi)}(x, y) = w_{(P_s,\pi)}(x - y)$$

defines a metric over $\mathbb{F}_q^n$ called the **poset block metric** or (P_s, π)**-metric**. The pair $(\mathbb{F}_q^n, d_{(P_s,\pi)})$ is said to be a **poset block space**. A k-dimensional $\mathbb{F}_q$-linear subspace C of V is said to be an $(n, k, d_{(P_s,\pi)})$ (P_s, π)**-code**, where $\mathbb{F}_q^n$ is equipped with the poset block metric $d_{(P_s,\pi)}(\cdot, \cdot)$ and

$$d_{(P_s,\pi)}(C) = \min\{w_{(P_s,\pi)}(c) : 0 \neq c \in C\}$$

is the (P_s, π)-minimum distance of C.

It is worth noting that when the label π satisfies $\pi(i) = 1\ \forall\ i \in [s]$, the poset block metric reduces to poset metric d_{P_n} introduced by Brualdi *et al.* [2]. Similarly, the block metric d_π introduced by Feng *et al.* [7] becomes a particular case of poset block metric when P_s is taken to be an anti-chain. When $\pi(i) = 1\,\forall i \in [s]$ and P_s is an anti-chain then the poset block metric reduces to the classical Hamming metric d_H.

For a metric d on V, the **ball** with center x and radius r is denoted by $B_d(x; r)$ and defined as

$$B_d(x; r) = \{y \in \mathbb{F}_q^n : d(x, y) \leq r\}.$$

The **packing radius** $R_d(C)$ **of a code** $C \subseteq V$ with respect to a metric d on V is defined as the greatest number r such that any two balls of radius r centered at distinct elements of C are disjoint.

Let $x \in \mathbb{F}_q^n$ and d be a metric over $\mathbb{F}_q^n$. The **packing radius of a vector** x is defined as the largest integer r satisfying

$$B_d(0; r) \cap B_d(x; r) = \phi$$

and is denoted by $R_d(x)$. By [4, Proposition 1], we have

$$R_d(C) = \min_{x \in C^*} R_d(x). \tag{2.1}$$

For a linear code $C \subseteq \mathbb{F}_q^n$ and a metric d over $\mathbb{F}_q^n$, a vector with the smallest packing radius is called a **packing vector** of the code.

Remark 2.1.

(1) Unlike the case of Hamming metric where $R_{d_H}(x) = \left\lfloor \frac{w_H(x)-1}{2} \right\rfloor$, the packing radius of a vector in the case of poset block metric follows the inequality:

$$\left\lfloor \frac{w_{(P_s,\pi)}(x) - 1}{2} \right\rfloor \leq R_{d_{(P_s,\pi)}}(x) \leq w_{(P_s,\pi)}(x) - 1.$$

(2) For $q = 2$, the packing radius of x is nothing but the packing radius of the code $C = \{0, x\}$.

(3) Finding the packing radius of a code corresponds to finding a packing vector of the code.

Let P_n be a poset and M_{P_n} be the set of its maximal elements. We define the **packing radius of the poset** P_n as

$$R(P_n) = \min_{A,B \subseteq M_{P_n}} \{\max\{|\langle A\rangle|, |\langle B\rangle|\}\} - 1$$

where (A, B) is a partition of M_{P_n}.

As proved in [4], the packing radius of a vector x is

$$R_{d_{P_n}}(x) = R(\langle \mathrm{supp}_H(x)\rangle) \tag{2.2}$$

where $\mathrm{supp}_H(x)$ denotes the Hamming support of the vector x, i.e., the set of all non-zero coordinate positions of x.

Now, we state the **poset partition problem** which is a generalization of the well-known NP-hard problem known as "the partition problem" [9]. Let P_n be a poset and (A, B) a partition of M_{P_n}, the maximal elements of P_n. We define the **discordancy** between A and B as

$$\Lambda(A, B) = \|\langle A\rangle| - |\langle B\rangle\| + |\langle A\rangle \cap \langle B\rangle|$$

and the **minimum discordancy** of P_n as

$$\Lambda^*(P_n) = \min_{X \sqcup Y = M_{P_n}} \Lambda(X, Y).$$

Remark 2.2. The maximum value of $\Lambda^*(P_n)$ is n and the minimum value is 0 or 1 depending on the parity of n.

By [4, Theorem 2], the packing radius of P_n can be written as

$$R(P_n) = \frac{n}{2} + \frac{\Lambda^*(P_n)}{2} - 1. \tag{2.3}$$

Remark 2.3. The packing radius of a poset P_n satisfies:

$$\left\lfloor \frac{n-1}{2} \right\rfloor \le R(P_n) \le n - 1.$$

As seen in [10, Theorem 5], the packing radius of an $(n, k, d_{(P_s,\pi)})$ NRT block code is

$$R_{d_{(P_s,\pi)}}(C) = d_{(P_s,\pi)}(C) - 1. \tag{2.4}$$

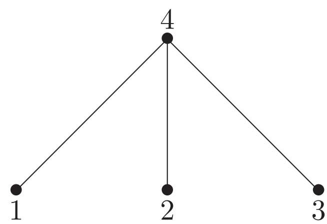
In [5, Definition 24], there is a discrepancy in the definition of subposet, may be typo or otherwise. However, the proofs of [5, Theorem 9] and [5, Proposition 6] following the definition still remain correct as there. We present the corrected version of the definition which is as follows:

Let $P = (A, \le_P)$ and $Q = (B, \le_Q)$ be two posets. We say P is a **subposet** of Q, denoted as $P \subseteq Q$, if $A \subseteq B$ and $x \le_P y \Rightarrow x \le_Q y\ \forall x, y \in A$.

3. The Packing Radius of a Code with Respect to Poset Block Metric

In this section, we extend a result on the packing radius with respect to poset metric given by D'Oliveira and Firer [4] to the case of poset block metric. Firstly, we provide an example to show that the lower bound of (1.1) can be attained when $d_{P_n} \neq d_H$, i.e., the poset is not an anti-chain.

Example 3.1. Consider $\mathbb{F}_3^4$ with the poset metric determined by the following poset



which is not an anti-chain.

Let C be the $(4, 2)$ poset code generated by the following generator matrix:

$$\begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

For the code C, $d_{P_4}(C) = 3$. Using the formula for the packing radius of a hierarchical poset code given in [5, Proposition 6] (also see [6, Proposition 3]), we get

$$R_{d_{P_4}}(C) = 3 + \frac{1-3}{2} - 1 = 1$$
$$= \left\lfloor \frac{d_{P_4} - 1}{2} \right\rfloor.$$

In the following example, we find that the upper bound of (1.1) has been attained even when the poset is not a chain.

Example 3.2. Consider $\mathbb{F}_2^4$ with the poset metric determined by the following poset

which is not a chain.

Let $C = \{0 = (0000), c_1 = (1100), c_2 = (0011), c_3 = (1111)\}$. For the code C, $d_{P_4}(C) = 2$. Now since the support of c_1 and c_2 are chains, using (2.4) we get

$$R_{d_{P_4}}(c_1) = R(\langle \mathrm{supp}_H(c_1) \rangle) = 1$$
$$\text{and} \quad R_{d_{P_4}}(c_2) = R(\langle \mathrm{supp}_H(c_2) \rangle) = 1.$$

Using (2.3), we get

$$R_{d_{P_4}}(c_3) = R(\langle \mathrm{supp}_H(c_3) \rangle) = \frac{4}{2} + \frac{0}{2} - 1 = 1.$$

Hence, using (2.1) we get

$$R_{d_{P_4}}(C) = \min_{x \in C^*} R_{d_{P_4}}(x) = 1 = d_{P_4}(C) - 1.$$

Remark 3.3. In view of Examples 3.1 and 3.2, we find that in (1.1), the lower bound is attained for any linear code if $d_{P_n} = d_H$ and the upper bound is attained for any linear code if the poset is a chain.

In [4, Theorem 1], the authors describe the packing radius of a vector with respect to poset metric. The following result extends it to the case of poset block metric. The proofs of lemmas and main theorem follow on similar lines as in the case of poset metric by visualizing $x \in \mathbb{F}_q^n$ as $(x_1 x_2 \dots x_s)$ where $x_i \in \mathbb{F}_q^{k_i}$ $\forall\ 1 \leq i \leq s$ and replacing support by π-support.

Theorem 3.4. *Let (P_s, π) be a poset block structure over $\mathbb{F}_q^n$ and $x \in \mathbb{F}_q^n$. Then*

$$R_{d_{(P_s,\pi)}}(x) = \min_{A,B \subseteq M_{supp_\pi(x)}} \{\max\{|\langle A\rangle|, |\langle B\rangle|\}\} - 1$$

where (A, B) is a partition of $M_{\mathrm{supp}_\pi(x)}$, the set of maximal elements of $\mathrm{supp}_\pi(x)$.

As a direct consequence of definition of packing radius of a poset, Theorem 3.4 yields:

Corollary 3.5. *The packing radius of a vector x is*

$$R_{d_{(P_s,\pi)}}(x) = R(\langle \mathrm{supp}_\pi(x)\rangle).$$

Remark 3.6. In the above, we deduce that as in the case of poset metric, the problem of finding the packing vector of a code in the case of poset block metric also reduces to "the poset partition problem". Further, the generalizations of Karp's differencing method and Korf's complete Karmarkar–Karp algorithm developed in [4] can now be applied to the case of poset block metric.

Now we discuss the packing radius of a code with respect to some particular kind of poset block metrics.

Case 1. P_s is a chain.
In this case,

$$\Lambda^*(P_s) = s.$$

Using (2.3), we get

$$R(P_s) = s - 1.$$

Let C be a (P_s, π)-code, i.e., an NRT block code. Since an ideal of a chain is itself a chain, therefore a minimum weight vector is a packing vector of the code C. Thus for an NRT block code C, the packing radius is given by

$$R_{d_{(P_s,\pi)}}(C) = d_{(P_s,\pi)}(C) - 1$$

as seen in [10].

Case 2. P_s is an anti-chain
In this case,

$$\Lambda^*(P_s) = \begin{cases} 1 & \text{if } s \text{ is odd,} \\ 0 & \text{if } s \text{ is even.} \end{cases}$$

Using (2.3), we get

$$R(P_s) = \left\lfloor \frac{s-1}{2} \right\rfloor.$$

Let C be a (P_s, π)-code, i.e., a linear error block code. Since an ideal of an anti-chain poset is itself an anti-chain, therefore a minimum weight vector is a packing vector of the code C. Thus, for a linear error block code C, the packing radius is given by

$$R_{d_\pi}(C) = \left\lfloor \frac{d_\pi(C) - 1}{2} \right\rfloor$$

as seen in [3].

Case 3. P_s is disjoint union of m chains of equal size r
In this case,

$$\Lambda^*(P_s) = \begin{cases} r & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even.} \end{cases}$$

Using (2.3), we get

$$R(P_s) = \begin{cases} \frac{s+r}{2} - 1 & \text{if } m \text{ is odd,} \\ \frac{s}{2} - 1 & \text{if } m \text{ is even.} \end{cases}$$

Since the ideal generated by support of any element may not be disjoint union of chains of equal size, we cannot give an explicit formula for the packing radius in this case. However, since the ideals generated by the maximal elements of the poset are disjoint, using [5, Theorem 3], we get that the poset partition problem in this case is equivalent to the classical partition problem.

We now extend the definition of shape of a vector [11] to the case of poset block metric. Let (P_s, π) be a poset block structure over $\mathbb{F}_q^n$, where P_s is disjoint union of m chains of length r:

$$[s] = U_1 \cup U_2 \cup \cdots \cup U_m, U_i \cap U_j = \phi (i \neq j);$$

$$u_{(i,j)} < u_{(i',j')} \text{ if and only if } i = i' \text{ and } j < j'.$$

Let $x \in \mathbb{F}_q^n$ be written as

$$x = (x_{11}, \ldots, x_{1r}; x_{21}, \ldots, x_{2r}; \ldots; x_{m1}, \ldots, x_{mr}).$$

The shape of x with respect to the poset block structure (P_s, π) is an $(r+1)$-vector $e = (e_0, e_1, \ldots, e_r)$ where

$$e_i = |\{U_j : \max(l : x_{jl} \neq 0) = i\}|, \quad \text{for } 1 \leq i \leq r \quad \text{and}$$

$$e_0 = m - \sum_{i=1}^{r} e_i.$$

For an ideal I of P_s, shape$(I) =$ shape(x) where

$$x_{ij} = \begin{cases} 1 & \text{if } u_{(i,j)} \in I, \\ 0 & \text{if } u_{(i,j)} \notin I. \end{cases}$$

Thus, shape$(x) =$ shape$(\langle \mathrm{supp}_\pi(x) \rangle)$.

Proposition 3.7. *When shape(x) = shape(y) with respect to (P_s, π) then $R_{d_{(P_s,\pi)}}(x) = R_{d_{(P_s,\pi)}}(y)$.*

Proof. The result follows from the fact that given the shape of a vector x in $\mathbb{F}_q^n$ with respect to (P_s, π),

$$\text{shape}(x) = (e_0, e_1, \ldots, e_r)$$

the packing radius of x with respect to (P_s, π) is equivalent to solving the classical partition problem for the multiset $\{\underbrace{e_i, e_i, \ldots, e_i}_{i \text{ times}} : i = 1, 2, \ldots, r\}$. □

We now provide a counter example to show that converse of Proposition 3.7 does not hold.

Example 3.8. Consider $\mathbb{F}_3^{30}$ with the poset block metric (P_{15}, π) where P_{15} is disjoint union of 3 chains of length 5 each and $\pi(i) = 2\, \forall 1 \le i \le 15$. Consider the following vectors in $\mathbb{F}_3^{30}$:

$$x = (12\,00\,00\,00\,00; 22\,11\,01\,12\,02; 00\,00\,00\,00\,00) \quad \text{and}$$

$$y = (00\,00\,00\,00\,00; 11\,21\,22\,10\,01; 01\,11\,10\,20\,22).$$

Using (2.3), we get

$$R_{d_{(P_s,\pi)}}(x) = \frac{6}{2} + \frac{4}{2} - 1 = 4 \quad \text{and}$$

$$R_{d_{(P_s,\pi)}}(y) = \frac{10}{2} - 1 = 4.$$

Thus, the packing radii of x and y are same but shape$(x) \neq$ shape(y) as

$$\text{shape}(x) = (1, 1, 0, 0, 0, 1) \quad \text{and}$$

$$\text{shape}(y) = (1, 0, 0, 0, 0, 2).$$

Case 4. P_s is a hierarchical poset with $M_{P_s} = \{x_1, x_2, \ldots, x_m\}$.
In this case, using the formula for the packing radius of a hierarchical poset determined in [5, Proposition 2], we get

$$\Lambda^*(P_s) = \begin{cases} s - m + 1 & \text{if } m \text{ is odd}, \\ s - m & \text{if } m \text{ is even}. \end{cases}$$

Using (2.3), we get

$$R(P_s) = \begin{cases} s + \frac{1-m}{2} - 1 & \text{if } m \text{ is odd,} \\ s - \frac{m}{2} - 1 & \text{if } m \text{ is even.} \end{cases}$$

In the following theorem, a generalization of [5, Proposition 6], the packing radius of a poset block code for hierarchical poset has been derived. The proof follows on the same lines and is therefore omitted.

Theorem 3.9. *Let (P_s, π) be a poset block structure where P_s is a hierarchical poset, $C \subseteq \mathbb{F}_q^n$ be a (P_s, π)-code and $c \in C$ be a minimum (P_s, π)-weight code-word. If I_c is the ideal generated by the π-support of c and $M_{I_c} = \{x_1, x_2, \ldots, x_m\}$ are the maximal elements of I_c, then*

$$R_{d_{(P_s,\pi)}}(C) = \begin{cases} w_{(P_s,\pi)}(C) + \frac{1-m}{2} - 1 & \textit{if } m \textit{ is odd,} \\ w_{(P_s,\pi)}(C) - \frac{m}{2} - 1 & \textit{if } m \textit{ is even.} \end{cases} \tag{3.1}$$

Proposition 3.10. *The packing radius of a code C with respect to a given poset block metric is always greater than or equal to that with respect to the corresponding block metric, i.e.,*

$$R_{d_{(P_s,\pi)}}(C) \geq R_{d_\pi}(C).$$

Proof. We know that for $P_s = [1]^s$, the poset block metric reduces to the corresponding block metric. We have

$$\begin{aligned} B_{(P_s,\pi)}(x; r) &= \{y \in \mathbb{F}_q^n : d_{(P_s,\pi)}(x, y) \leq r\} \\ &= \{y \in \mathbb{F}_q^n : w_{(P_s,\pi)}(x - y) \leq r\}. \end{aligned}$$

Since $w_\pi(x) \leq w_{(P_s,\pi)}(x), \forall\, x \in \mathbb{F}_q^n$, we have

$$\begin{aligned} \{y \in \mathbb{F}_q^n : w_{(P_s,\pi)}(x - y) \leq r\} &\subseteq \{y \in \mathbb{F}_q^n : w_\pi(x - y) \leq r\} \\ &= \{y \in \mathbb{F}_q^n : d_\pi(x, y) \leq r\} \\ &= B_\pi(x; r) \\ \text{i.e., } B_{(P_s,\pi)}(x; r) &\subseteq B_\pi(x; r). \end{aligned}$$

Hence the result. □

Remark 3.11. We know that poset block metric cannot be reduced to the corresponding poset metric directly. But, we may get a non-linear poset code as the image of a linear poset-block code by considering a map as follows:

$$\begin{aligned} f : (\mathbb{F}_q^n, d_{(P_s,\pi)}) &\to (\mathbb{F}_2^s, d_{P_s}) \\ (x_1 x_2 \ldots x_s) &\mapsto (\delta_{x_1} \delta_{x_2} \ldots \delta_{x_s}) \end{aligned}$$

$$\text{where } \delta_{x_i} = \begin{cases} 1 & \text{if } x_i \neq 0, \\ 0 & \text{if } x_i = 0. \end{cases}$$

$$\text{Then } \operatorname{supp}_\pi(x) = \operatorname{supp}_H(f(x)).$$

Such a map is weight preserving but not linear.

Considering the packing radius, we note that

$$\begin{aligned} R_{d_{(P_s,\pi)}}(x) &= R(\langle \operatorname{supp}_\pi(x) \rangle) \\ &= R(\langle \operatorname{supp}_H(f(x)) \rangle) \\ &= R_{d_{P_s}}(f(x)). \end{aligned}$$

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