

Characteristic properties of σ -A-nuclei of a quasigroup

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In this paper, we investigate the properties of σ -A-nuclei of a quasigroup including relations between them and relations between their respective component sets, where $\sigma \in S_3$. We also find connections between components of σ -A-nuclei of a quasigroup and components of σ -A-nuclei of the isotrophic images. Further, we investigate the properties of various inverse quasigroups using the derived connections. These properties will not only make the study of σ -A-nuclei of a quasigroup simple but also reduce the time required in the computation of σ -A-nuclei of a quasigroup for different values of σ .

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1. Introduction

Garrison [8] introduced the notion of quasigroup nuclei in 1940. It measures how far a quasigroup is from a group by measuring the near-associativity of the quasigroup. It has been shown in [16] that if a quasigroup $(Q, \cdot)$ has a non-trivial left/ right/

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middle nucleus then the quasigroup $(Q, \cdot)$ has a left identity/ a right identity/ an identity element. A detailed historical background of A-nuclei of a quasigroup has been discussed by Shcherbacov in [19, 20]. Belousov [1] generalized Garrison's nuclei to proper quasigroups by introducing the concept of left, right, and middle regular permutations, which can be used to measure the quasigroups' near-associativity in the absence of a left or right unit. Several authors have investigated connections between quasigroup nuclei and groups of regular permutations of quasigroups [1, 2, 4, 14]. The A-nuclei (autotopy nuclei) of a quasigroup are then identified as groups of left, right, and middle regular permutations, and have been studied extensively by Belousov, Kepka, Keedwell, Anthony, Shcherbacov, etc. [1, 11, 14, 19]. It is worth noting that if the quasigroup is a loop, then the non-trivial components of the loop's left/right A-nuclei coincide with left/right translations by elements of the left/right nuclei, and hence the notion of A-nuclei may be considered as a generalization of a quasigroup's nuclei. The connections between components of the A-nuclei of a quasigroup and components of the A-nuclei of its isotrophic images have been discussed in [20]. The reader may refer [19, 20] for a detailed survey on A-nuclei of a quasigroup.

The concept of left, right and middle σ -A-nuclei of a quasigroup, as a generalization of left, right and middle A-nuclei respectively, has been introduced in [7]. In this paper we characterize the inverse sets of σ -A nuclei and their respective components. Further, characterization of the products of σ -A-nuclei and τ -A-nuclei of a quasigroup and their respective component sets have been discussed. We also find connections between σ -A-nuclei of a quasigroup and σ -A-nuclei of its isotrophic images, as well as between their respective component sets. Further, we use the connections of σ -A-nucleus of a quasigroup and its isotrophic images to investigate properties of σ -A-nuclei of various inverse quasigroups.

The paper has the following structure: Section 2 presents some basic definitions, notations and results required for the subsequent sections. In Sec. 3, we define σ -A-nuclei of a quasigroup and investigate some of their properties. Further, we derive connections between components of the σ -A-nuclei of a quasigroup and its isotrophic images. Section 4 discusses properties of various inverse quasigroups. Conclusions are finally drawn in Sec. 5.

2. Preliminaries

In this section, we present some definitions and notations required for our study in this paper [4, 6, 16–20].

Definition 2.1. A groupoid $(Q, *)$ is called a *quasigroup* if there exist unique solutions $x, y \in Q$ of the equations $x * a = b$ and $a * y = b$ for all ordered pair $(a, b) \in Q^2$.

In other words, a groupoid $(Q, *)$ is called a quasigroup if in the equality $x*y = z$, knowing any two elements from x, y, z uniquely specifies the remaining one element.

In view of the above definition, given a quasigroup $(Q, *)$ it is possible to associate five other operators $*^{(12)}, *^{(13)}, *^{(23)}, *^{(123)}$ and $*^{(132)}$, known as *parastrophes* of quasigroup $(Q, *)$ as follows:

$$\begin{aligned} x * y = z &\iff y *^{(12)} x = z \iff z *^{(13)} y = x \iff x *^{(23)} z = y \\ &\iff y *^{(123)} z = x \iff z *^{(132)} x = y. \end{aligned}$$

Note that the quasigroup $(Q, *^\tau)$ is known as τ -parastrophe or τ -parastrophic image of quasigroup $(Q, *)$, where $\tau \in S_3$.

Let $(Q, *)$ be a groupoid and let c be a fixed element of Q . The maps $L_c: Q \rightarrow Q$ and $R_c: Q \rightarrow Q$ defined as $L_c x = c * x$ and $R_c x = x * c$, for all $x \in Q$, are, respectively, called the *left and the right translations*.

If $(Q, *)$ is a quasigroup then it is possible to define third kind of translation, known as *middle translation*, $P_c: Q \rightarrow Q$ defined as $x * P_c x = c$, for all $x \in Q$.

In terms of translation maps Definition 2.1 can also be written as: A groupoid $(Q, *)$ is called a quasigroup if the left and right translations L_c and R_c are bijective maps for all $c \in Q$.

Definition 2.2. A groupoid $(G, \circ)$ is an *isotopic image* or *isotope* of a groupoid $(G, *)$, if there exist permutations $\alpha_1, \alpha_2, \alpha_3$ of the set G such that

$$x \circ y = \alpha_3^{-1}(\alpha_1 x * \alpha_2 y) \quad (2.1)$$

for all $x, y \in G$.

We can also write the equality (2.1) as $(G, \circ) = (G, *)R$, where the triplet $R = (\alpha_1, \alpha_2, \alpha_3)$ is called an isotopism or isotopy of the groupoid $(G, *)$. It can easily be seen that an isotopic image of a quasigroup is also a quasigroup.

If the two operators $\circ$ and $*$ are equal then triplet R is called an *autotopism* or *autotopy* of binary groupoid $(G, *)$. Let $Aut(G, *)$ denote the set of all autotopies of a groupoid $(G, *)$. It can easily be seen that $Aut(G, *)$ forms a group with respect to usual component-wise multiplications of autotopies.

If $\tau \in S_3$ and $R = (\alpha_1, \alpha_2, \alpha_3)$ is an isotopy of a binary groupoid $(Q, *)$, then the action of τ on the triplet R denoted by R^τ shall be defined as $R^\tau = (\alpha_{\tau^{-1}1}, \alpha_{\tau^{-1}2}, \alpha_{\tau^{-1}3})$.

Definition 2.3. A quasigroup (Q, B) is an *isostrophic image* or *isostrophe* of (Q, A) , if there exists a collection of permutations $(\sigma, (\alpha_1, \alpha_2, \alpha_3)) = (\sigma, R)$, where $\sigma \in S_3$ and $R = (\alpha_1, \alpha_2, \alpha_3)$ is triplet of permutations $\alpha_1, \alpha_2, \alpha_3$ of the set Q such that

$$B(x_1, x_2) = A(x_1, x_2)(\sigma, R) = \alpha_3^{-1}A(\alpha_1 x_{\sigma^{-1}1}, \alpha_2 x_{\sigma^{-1}2}) \quad (2.2)$$

for all $x_1, x_2 \in Q$.

The tuple (σ, R) is known as *isostrophism* or *isostrophy* of the quasigroup (Q, A) . We can rewrite (2.2) as $B = (A^\sigma)R$. We shall call α_i , the i th component of the isostrophism (σ, R) , for $i = 1, 2, 3$. Definition 2.3 can also be written as: An isostrophic image of a quasigroup is defined as an isotopic image of its parastrophe.

Let (σ, R) and (τ, S) be isotrophisms of a quasigroup (Q, A) . Then their multiplication is defined as follows:

$$(\sigma, R)(\tau, S) = (\sigma\tau, R^\tau S), \quad (2.3)$$

where $A^{\sigma\tau} = (A^\sigma)^\tau$ and $(x_1, x_2, x_3)(R^\tau S) = ((x_1, x_2, x_3)R^\tau)S$, for all quasigroup triplets (x_1, x_2, x_3) .

It may be noted that this multiplication has been defined in different way by Belousov, Lyakh, Keedwell and Shcherbacov [3, 13, 15].

The inverse of an isotrophism (σ, R) is

$$(\sigma, R)^{-1} = (\sigma^{-1}, (R^{-1})^{\sigma^{-1}}) = (\sigma^{-1}, (\alpha_{\sigma 1}^{-1}, \alpha_{\sigma 2}^{-1}, \alpha_{\sigma 3}^{-1})). \quad (2.4)$$

If the binary operations A and B in Definition 2.3 are equal then the tuple (σ, R) is called an *autostrophism* or *autostrophy* of the quasigroup (Q, A) . Let $\text{Aus}(Q, A)$ denote the set of all autostrophisms of quasigroup (Q, A) . $\text{Aus}(Q, A)$ forms a group under the multiplication operation defined in (2.3).

Theorem 2.1 ([4]). *If quasigroup $(Q, \circ)$ is an isotrophic image of quasigroup $(Q, *)$ with an isotrophy θ , i.e. $(Q, \circ) = (Q, *)\theta$, then $\text{Aus}(Q, \circ) = \theta^{-1}\text{Aus}(Q, *)\theta$.*

Definition 2.4. Let $(G, *)$ be a groupoid and let c be an element of G . Then c is left (right, middle) nuclear in $(G, *)$ if $L_{c*x} = L_x L_c$ ($R_{x*c} = R_x R_c$, $L_{x*c} = L_c L_x$) for all $x \in G$.

If c is left, right and middle nuclear in $(G, *)$ then c is called nuclear in groupoid $(G, *)$.

Definition 2.5. The left nucleus N_l (right nucleus N_r , middle nucleus N_m) of a quasigroup $(Q, *)$ is the set of all left (right, middle) nuclear elements in $(Q, *)$. Equivalently,

$$N_l = \{a \in Q \mid a * (x * y) = (a * x) * y, \forall x, y \in Q\},$$

$$N_r = \{a \in Q \mid (x * y) * a = x * (y * a), \forall x, y \in Q\}$$

and

$$N_m = \{a \in Q \mid (x * a) * y = x * (a * y), \forall x, y \in Q\}.$$

The nucleus N of the quasigroup $(Q, *)$ is defined as $N = N_l \cap N_r \cap N_m$.

If N_l (N_r , N_m) is non-empty, then N_l (N_r , N_m) is a subgroup of quasigroup $(Q, *)$ [16]. It has been shown that there is a weakness in Garrison's nucleus, viz., if a quasigroup $(Q, *)$ has a non-trivial (or non-empty) left or right or middle nucleus then the quasigroup is a left loop or a right loop or a loop, respectively [8, 16].

Definition 2.6. The left (right, middle) A-nucleus of a quasigroup $(Q, *)$ is defined as the set of all autotopisms of the form $(\alpha, \varepsilon, \gamma)$ ($(\varepsilon, \beta, \gamma)$, $(\alpha, \beta, \varepsilon)$) of the quasigroup $(Q, *)$, where α, β, γ are permutations of the set Q and ε is the identity mapping.

Table 1. Connections between components of A-nuclei of a quasigroup and its isostrophic images.

	(ε, R)	$((1\ 2), R)$	$((1\ 3), R)$	$((2\ 3), R)$	$((1\ 3\ 2), R)$	$((1\ 2\ 3), R)$
${}_1N_l^\circ$	$\alpha^{-1}{}_1N_l\alpha$	$\alpha^{-1}{}_2N_r\alpha$	$\alpha^{-1}{}_3N_l\alpha$	$\alpha^{-1}{}_1N_m\alpha$	$\alpha^{-1}{}_2N_m\alpha$	$\alpha^{-1}{}_3N_r\alpha$
${}_3N_l^\circ$	$\gamma^{-1}{}_3N_l\gamma$	$\gamma^{-1}{}_3N_r\gamma$	$\gamma^{-1}{}_1N_l\gamma$	$\gamma^{-1}{}_2N_m\gamma$	$\gamma^{-1}{}_1N_m\gamma$	$\gamma^{-1}{}_2N_r\gamma$
${}_2N_r^\circ$	$\beta^{-1}{}_2N_r\beta$	$\beta^{-1}{}_1N_l\beta$	$\beta^{-1}{}_2N_m\beta$	$\beta^{-1}{}_3N_r\beta$	$\beta^{-1}{}_3N_l\beta$	$\beta^{-1}{}_1N_m\beta$
${}_3N_r^\circ$	$\gamma^{-1}{}_3N_r\gamma$	$\gamma^{-1}{}_3N_l\gamma$	$\gamma^{-1}{}_1N_m\gamma$	$\gamma^{-1}{}_2N_r\gamma$	$\gamma^{-1}{}_1N_l\gamma$	$\gamma^{-1}{}_2N_m\gamma$
${}_1N_m^\circ$	$\alpha^{-1}{}_1N_m\alpha$	$\alpha^{-1}{}_2N_m\alpha$	$\alpha^{-1}{}_3N_r\alpha$	$\alpha^{-1}{}_1N_l\alpha$	$\alpha^{-1}{}_2N_r\alpha$	$\alpha^{-1}{}_3N_l\alpha$
${}_2N_m^\circ$	$\beta^{-1}{}_2N_m\beta$	$\beta^{-1}{}_1N_m\beta$	$\beta^{-1}{}_2N_r\beta$	$\beta^{-1}{}_3N_l\beta$	$\beta^{-1}{}_3N_r\beta$	$\beta^{-1}{}_1N_l\beta$

Note that the symbol A in the above definition stands for autotopy. We shall, respectively, denote these three sets of autotopisms by N_l^A , N_r^A and N_m^A and their sets of all i th components by ${}_iN_l^A$, ${}_iN_r^A$ and ${}_iN_m^A$, respectively, for $i = 1, 2, 3$.

Table 1 shows connections between components of the A-nuclei of a quasigroup $(Q, \cdot)$ and components of the A-nuclei of its isostrophic images of the form $(Q, \circ) = (Q, \cdot)(\sigma, R)$, where $\sigma \in S_3$, $R = (\alpha, \beta, \gamma)$ and α, β, γ are permutations of the set Q [19]. Note that, to denote the A-nuclei of quasigroups $(Q, \circ)$ and $(Q, \cdot)$ the symbol A (for autotopy) is replaced by the binary operations $\circ$ and $\cdot$, respectively.

The left, right and middle A-nuclei of a quasigroup have been generalized to the left, right and middle σ -A-nuclei respectively, for $\sigma \in S_3$, as follows:

Definition 2.7 ([7]). The left (right, middle) σ -A-nuclei of a quasigroup $(Q, \cdot)$ is defined as the set of all autostrophisms of the form $(\sigma, (\alpha, \varepsilon, \gamma))$ $((\sigma, (\varepsilon, \beta, \gamma)), (\sigma, (\alpha, \beta, \varepsilon)))$ of the quasigroup $(Q, \cdot)$, where α, β, γ are permutations of the set Q , ε is the identity mapping and $\sigma \in S_3$.

Note that, the left, right and middle ε -A-nuclei of a quasigroup can be considered as the left, right and middle A-nuclei of the quasigroup, respectively.

We shall denote the left, right and middle σ -A-nuclei by ${}^\sigma N_l^A$, ${}^\sigma N_r^A$ and ${}^\sigma N_m^A$, respectively. Let ${}^\sigma {}_iN_l^A$, ${}^\sigma {}_iN_r^A$ and ${}^\sigma {}_iN_m^A$ denote the respective sets of all i th components of members of the left, right and middle σ -A-nuclei, for $i = 1, 2, 3$.

It may be noted that ${}^\sigma {}_2N_l^A = {}^\sigma {}_1N_r^A = {}^\sigma {}_3N_m^A = \{\varepsilon\}$. We shall call these sets as trivial component sets.

Remark 2.1. It may be noted that $(\sigma, (\alpha_1, \alpha_2, \alpha_3)) \in {}^\sigma N_l^A$ if and only if $(\sigma^{-1}, (\alpha_{\sigma^{-1}1}, \alpha_{\sigma^{-1}2}, \alpha_{\sigma^{-1}3})) \in {}^{\sigma^{-1}} N_v^A$, where

$$v = \begin{cases} r & \text{if } \sigma^{-1}2 = 1, \\ l & \text{if } \sigma^{-1}2 = 2, \\ m & \text{if } \sigma^{-1}2 = 3. \end{cases}$$

Thus we have

$${}^\sigma N_l^A \neq \emptyset \text{ iff } {}^{\sigma^{-1}} N_v^A \neq \emptyset, \quad \text{for all } \sigma \in S_3. \quad (2.5)$$

3. Properties of σ -A-Nuclei

In this section we characterize the inverse sets of σ -A-nuclei and the products of two σ -A-nuclei of a quasigroup, and their respective component sets. Further, we find connections between σ -A-nuclei of a quasigroup and σ -A-nuclei of its isostrophic images and parastrophic images, and their respective component sets.

Recall that if G is a group and $B \subseteq G$ is any subset, then we denote by B^{-1} the set of inverses of elements of B , i.e. $B^{-1} = \{x^{-1} \mid x \in B\}$. Let $(Q, \cdot)$ be a quasigroup and $G = \text{Aus}(Q, \cdot)$, the group all autostrophies of $(Q, \cdot)$. For each permutation $\sigma \in S_3$, we shall characterize B^{-1} , where $B = {}^\sigma N_v^A \subseteq \text{Aus}(Q, \cdot)$, for $v \in \{l, m, r\}$. For more clarity in notations and result, let us consider a particular example.

Let $\sigma = (1\ 2)$ and $v = r$. We claim that

$$({}^{(1\ 2)}N_r^A)^{-1} = {}^{(1\ 2)}N_l^A. \quad (3.1)$$

In view of Remark 2.1, the result is trivial for the case ${}^{(1\ 2)}N_r^A = \emptyset$. So, we prove (3.1) for ${}^{(1\ 2)}N_r^A \neq \emptyset$. We shall show the inclusions in both directions. Let $\varphi \in ({}^{(1\ 2)}N_r^A)^{-1}$. Then we have $\varphi^{-1} \in {}^{(1\ 2)}N_r^A$. Thus there exist permutations α_2, α_3 of Q such that $\varphi^{-1} = ((1\ 2), (\varepsilon, \alpha_2, \alpha_3))$, which implies

$$\varphi = ((1\ 2), (\alpha_2^{-1}, \varepsilon, \alpha_3^{-1})) \in {}^{(1\ 2)}N_l^A. \quad (3.2)$$

Thus $({}^{(1\ 2)}N_r^A)^{-1} \subseteq {}^{(1\ 2)}N_l^A$. The other way inclusion can be shown on similar lines.

Also, from (3.2) we have $\alpha_2^{-1} \in {}^{(1\ 2)}N_l^A$ and $\alpha_3^{-1} \in {}^{(1\ 2)}N_l^A$, which implies $({}^{(1\ 2)}N_r^A)^{-1} \subseteq {}^{(1\ 2)}N_l^A$ and $({}^{(1\ 2)}N_r^A)^{-1} \subseteq {}^{(1\ 2)}N_l^A$. Similarly, ${}^{(1\ 2)}N_l^A \subseteq ({}^{(1\ 2)}N_r^A)^{-1}$ and ${}^{(1\ 2)}N_l^A \subseteq ({}^{(1\ 2)}N_r^A)^{-1}$. Thus

$$({}^{(1\ 2)}N_r^A)^{-1} = {}^{(1\ 2)}N_l^A \quad \text{and} \quad ({}^{(1\ 2)}N_l^A)^{-1} = {}^{(1\ 2)}N_r^A.$$

Note that we shall omit the relations for the trivial component sets.

The following theorem gives a complete description of the inverse sets of σ -A-nuclei of a quasigroup and their respective component sets, for any $\sigma \in S_3$.

Theorem 3.1. *If $(Q, \cdot)$ is a quasigroup, then for all $\sigma \in S_3$*

(1) $({}^\sigma N_l^A)^{-1} = {}^{\sigma^{-1}}N_v^A$, where

$$v = \begin{cases} r & \text{if } \sigma^{-1}2 = 1, \\ l & \text{if } \sigma^{-1}2 = 2, \\ m & \text{if } \sigma^{-1}2 = 3. \end{cases}$$

Further, $({}^\sigma N_l^A)^{-1} = {}^{\sigma^{-1}}N_v^A$ and $({}^\sigma N_l^A)^{-1} = {}^{\sigma^{-1}}N_v^A$.

(2) $({}^\sigma N_r^A)^{-1} = {}^{\sigma^{-1}}N_v^A$, where

$$v = \begin{cases} r & \text{if } \sigma^{-1}1 = 1, \\ l & \text{if } \sigma^{-1}1 = 2, \\ m & \text{if } \sigma^{-1}1 = 3. \end{cases}$$

Further, $(\sigma N_r^A)^{-1} = \sigma_{\sigma^{-1}2}^{-1} N_v^A$ and $(\sigma N_r^A)^{-1} = \sigma_{\sigma^{-1}3}^{-1} N_v^A$.

(3) $(\sigma N_m^A)^{-1} = \sigma^{-1} N_v^A$, where

$$v = \begin{cases} r & \text{if } \sigma^{-1}3 = 1, \\ l & \text{if } \sigma^{-1}3 = 2, \\ m & \text{if } \sigma^{-1}3 = 3. \end{cases}$$

Further, $(\sigma N_m^A)^{-1} = \sigma_{\sigma^{-1}1}^{-1} N_v^A$ and $(\sigma N_m^A)^{-1} = \sigma_{\sigma^{-1}2}^{-1} N_v^A$.

Proof. (1) From Remark 2.1 it is clear that $\sigma N_l^A = \emptyset$ if and only if $\sigma^{-1} N_v^A = \emptyset$, so we are done. We prove the result for $\sigma N_l^A \neq \emptyset$. We shall show the inclusions in both directions. Let $\varphi \in (\sigma N_l^A)^{-1}$. Then $\varphi^{-1} \in \sigma N_l^A$. Thus there exist permutations $\alpha_1, \alpha_2, \alpha_3$ of Q such that $\varphi^{-1} = (\sigma, (\alpha_1, \alpha_2, \alpha_3))$, where $\alpha_2 = \varepsilon$. Hence

$$\varphi = (\varphi^{-1})^{-1} = (\sigma^{-1}, (\alpha_{\sigma 1}^{-1}, \alpha_{\sigma 2}^{-1}, \alpha_{\sigma 3}^{-1})). \quad (3.3)$$

Let $\varphi = (\sigma^{-1}, (\varphi_1, \varphi_2, \varphi_3))$. On comparing the components we get $\varphi_i = \alpha_{\sigma i}^{-1}$, i.e.

$$\varphi_{\sigma^{-1}i} = \alpha_i^{-1}, \quad \text{for } i = 1, 2, 3. \quad (3.4)$$

When $i = 2$, $\varphi_{\sigma^{-1}2} = \alpha_2^{-1} = \varepsilon$, which implies that $(\sigma^{-1}2)$ th component of φ is identity. This shows $\varphi \in \sigma_{\sigma^{-1}2}^{-1} N_v^A$ and hence $(\sigma N_l^A)^{-1} \subseteq \sigma_{\sigma^{-1}2}^{-1} N_v^A$. Also, from (3.4) and $\varphi = (\sigma^{-1}, (\varphi_1, \varphi_2, \varphi_3)) \in \sigma_{\sigma^{-1}i}^{-1} N_v^A$ we get $\alpha_i^{-1} \in \sigma_{\sigma^{-1}i}^{-1} N_v^A$, which implies $(\sigma N_l^A)^{-1} \subseteq \sigma_{\sigma^{-1}i}^{-1} N_v^A$, for $i = 1, 3$.

Conversely, let $\varphi \in \sigma_{\sigma^{-1}i}^{-1} N_v^A$. Then there exist permutations $\varphi_1, \varphi_2, \varphi_3$ of Q such that $\varphi = (\sigma^{-1}, (\varphi_1, \varphi_2, \varphi_3))$, where $\varphi_{\sigma^{-1}2} = \varepsilon$. Thus

$$\varphi^{-1} = (\sigma, (\varphi_{\sigma^{-1}1}^{-1}, \varphi_{\sigma^{-1}2}^{-1}, \varphi_{\sigma^{-1}3}^{-1})) = (\sigma, (\varphi_{\sigma^{-1}1}^{-1}, \varepsilon, \varphi_{\sigma^{-1}3}^{-1})) \in \sigma N_l^A. \quad (3.5)$$

This shows $\varphi \in (\sigma N_l^A)^{-1}$ and hence $\sigma_{\sigma^{-1}i}^{-1} N_v^A \subseteq (\sigma N_l^A)^{-1}$. Also, from (3.5) we get $\varphi_{\sigma^{-1}i} \in (\sigma_i N_l^A)^{-1}$, which gives $\sigma_{\sigma^{-1}1}^{-1} N_v^A \subseteq (\sigma_1 N_l^A)^{-1}$, for $i = 1, 3$ (since $\varphi_i \in \sigma_{\sigma^{-1}i}^{-1} N_v^A$).

(2) and (3) can be proved on similar lines. $\square$

Using Theorem 3.1 the components of inverse sets of a left, right and middle σ -A-nuclei can be found from the components of left, right and middle σ^{-1} -A-nuclei of a quasigroup and vice versa. Table 2 shows relationships between components of σ -A-nuclei and σ^{-1} -A-nuclei of a quasigroup $(Q, \cdot)$ obtained from Theorem 3.1, for all $\sigma \in S_3$.

Recall that if G is a group and B and C are subsets of G , then the product of B and C is the subset of G defined as

$$BC = \{bc \mid b \in B \text{ and } c \in C\}.$$

It may be noted that $BC = \emptyset$ if and only if at least one of $B, C = \emptyset$.

Let $(Q, \cdot)$ be a quasigroup and $G = \text{Aus}(Q, \cdot)$. For each permutation $\sigma, \tau \in S_3$, we shall characterize BC , where $B = \sigma N_u^A \subseteq \text{Aus}(Q, \cdot)$ and $C = \tau N_v^A \subseteq \text{Aus}(Q, \cdot)$, where $u, v \in \{l, m, r\}$.

Table 2. Relationships between components of σ -A-nuclei and σ^{-1} -A-nuclei.

σ			
ε	$({}_1N_l^A)^{-1} = {}_1N_l^A$	$({}_2N_r^A)^{-1} = {}_2N_r^A$	$({}_1N_m^A)^{-1} = {}_1N_m^A$
	$({}_3N_l^A)^{-1} = {}_3N_l^A$	$({}_3N_r^A)^{-1} = {}_3N_r^A$	$({}_2N_m^A)^{-1} = {}_2N_m^A$
(12)	$({}^{(12)}{}_1N_l^A)^{-1} = ({}^{(12)}{}_2N_r^A$	$({}^{(12)}{}_2N_r^A)^{-1} = ({}^{(12)}{}_1N_l^A$	$({}^{(12)}{}_2N_m^A)^{-1} = ({}^{(12)}{}_1N_m^A$
	$({}^{(12)}{}_3N_l^A)^{-1} = ({}^{(12)}{}_3N_r^A$	$({}^{(12)}{}_3N_r^A)^{-1} = ({}^{(12)}{}_3N_l^A$	$({}^{(12)}{}_1N_m^A)^{-1} = ({}^{(12)}{}_2N_m^A$
(13)	$({}^{(13)}{}_3N_l^A)^{-1} = ({}^{(13)}{}_1N_l^A$	$({}^{(13)}{}_3N_r^A)^{-1} = ({}^{(13)}{}_1N_m^A$	$({}^{(13)}{}_2N_m^A)^{-1} = ({}^{(13)}{}_2N_r^A$
	$({}^{(13)}{}_1N_l^A)^{-1} = ({}^{(13)}{}_3N_l^A$	$({}^{(13)}{}_2N_r^A)^{-1} = ({}^{(13)}{}_2N_m^A$	$({}^{(13)}{}_1N_m^A)^{-1} = ({}^{(13)}{}_3N_r^A$
(23)	$({}^{(23)}{}_1N_l^A)^{-1} = ({}^{(23)}{}_1N_m^A$	$({}^{(23)}{}_3N_r^A)^{-1} = ({}^{(23)}{}_2N_r^A$	$({}^{(23)}{}_1N_m^A)^{-1} = ({}^{(23)}{}_1N_l^A$
	$({}^{(23)}{}_3N_l^A)^{-1} = ({}^{(23)}{}_2N_m^A$	$({}^{(23)}{}_2N_r^A)^{-1} = ({}^{(23)}{}_3N_r^A$	$({}^{(23)}{}_2N_m^A)^{-1} = ({}^{(23)}{}_3N_l^A$
(123)	$({}^{(123)}{}_3N_l^A)^{-1} = ({}^{(132)}{}_2N_r^A$	$({}^{(123)}{}_2N_r^A)^{-1} = ({}^{(132)}{}_1N_m^A$	$({}^{(123)}{}_2N_m^A)^{-1} = ({}^{(132)}{}_1N_l^A$
	$({}^{(123)}{}_1N_l^A)^{-1} = ({}^{(132)}{}_3N_r^A$	$({}^{(123)}{}_3N_r^A)^{-1} = ({}^{(132)}{}_2N_m^A$	$({}^{(123)}{}_1N_m^A)^{-1} = ({}^{(132)}{}_3N_l^A$
(132)	$({}^{(132)}{}_3N_l^A)^{-1} = ({}^{(123)}{}_1N_m^A$	$({}^{(132)}{}_3N_r^A)^{-1} = ({}^{(123)}{}_1N_l^A$	$({}^{(132)}{}_1N_m^A)^{-1} = ({}^{(123)}{}_2N_r^A$
	$({}^{(132)}{}_1N_l^A)^{-1} = ({}^{(123)}{}_2N_m^A$	$({}^{(132)}{}_2N_r^A)^{-1} = ({}^{(123)}{}_3N_l^A$	$({}^{(132)}{}_2N_m^A)^{-1} = ({}^{(123)}{}_3N_r^A$

The following theorem gives a complete description of products of σ -A-nuclei and τ -A-nuclei of a quasigroup and their respective component sets, for any $\sigma, \tau \in S_3$.

Theorem 3.2. *If $(Q, \cdot)$ is a quasigroup, then for all $\sigma, \tau \in S_3$*

(1) $\sigma N_v^A \tau N_l^A = \sigma\tau N_l^A$, where

$$v = \begin{cases} r & \text{if } \tau^{-1}2 = 1, \\ l & \text{if } \tau^{-1}2 = 2, \\ m & \text{if } \tau^{-1}2 = 3, \end{cases}$$

provided $\sigma N_v^A, \tau N_l^A \neq \emptyset$. Further, ${}_{\tau^{-1}1}\sigma N_v^A {}_{\tau^{-1}1}\tau N_l^A = {}_{\tau^{-1}1}\sigma\tau N_l^A$ and ${}_{\tau^{-1}3}\sigma N_v^A {}_{\tau^{-1}3}\tau N_l^A = {}_{\tau^{-1}3}\sigma\tau N_l^A$.

(2) $\sigma N_v^A \tau N_r^A = \sigma\tau N_r^A$, where

$$v = \begin{cases} r & \text{if } \tau^{-1}1 = 1, \\ l & \text{if } \tau^{-1}1 = 2, \\ m & \text{if } \tau^{-1}1 = 3, \end{cases}$$

provided $\sigma N_v^A, \tau N_r^A \neq \emptyset$. Further, ${}_{\tau^{-1}2}\sigma N_v^A {}_{\tau^{-1}2}\tau N_r^A = {}_{\tau^{-1}2}\sigma\tau N_r^A$ and ${}_{\tau^{-1}3}\sigma N_v^A {}_{\tau^{-1}3}\tau N_r^A = {}_{\tau^{-1}3}\sigma\tau N_r^A$.

(3) $\sigma N_v^A \tau N_m^A = \sigma\tau N_m^A$, where

$$v = \begin{cases} r & \text{if } \tau^{-1}3 = 1, \\ l & \text{if } \tau^{-1}3 = 2, \\ m & \text{if } \tau^{-1}3 = 3, \end{cases}$$

provided ${}^\sigma N_v^A, {}^\tau N_m^A \neq \emptyset$. Further, ${}_{\tau^{-1}1}{}^\sigma N_v^A {}^\tau N_m^A = {}^{\sigma\tau} N_m^A$ and ${}_{\tau^{-1}2}{}^\sigma N_v^A {}^\tau N_m^A = {}^{\sigma\tau} N_m^A$.

Proof. (1) We shall show the inclusions in both directions. Let $\phi \in {}^\sigma N_v^A$ and $\psi \in {}^\tau N_l^A$. Then there exist permutations ϕ_1, ϕ_2, ϕ_3 and ψ_1, ψ_3 of Q such that $\phi = (\sigma, (\phi_1, \phi_2, \phi_3))$ and $\psi = (\tau, (\psi_1, \varepsilon, \psi_3))$, where $\phi_{\tau^{-1}2} = \varepsilon$. Then

$$\phi\psi = (\sigma\tau, (\phi_{\tau^{-1}1}\psi_1, \phi_{\tau^{-1}2}, \phi_{\tau^{-1}3}\psi_3)) \quad (3.6)$$

$$= (\sigma\tau, (\phi_{\tau^{-1}1}\psi_1, \varepsilon, \phi_{\tau^{-1}3}\psi_3)) \in {}^{\sigma\tau} N_l^A \quad (3.7)$$

(because $\phi\psi \in \text{Aus}(Q, \cdot)$). Thus ${}^\sigma N_v^A {}^\tau N_l^A \subseteq {}^{\sigma\tau} N_l^A$. This gives ${}^{\sigma\tau} N_l^A \neq \emptyset$, since ${}^\sigma N_v^A, {}^\tau N_l^A \neq \emptyset$. Also from (3.7) we obtain $\phi_{\tau^{-1}1}\psi_1 \in {}^{\sigma\tau} N_l^A$ and $\phi_{\tau^{-1}3}\psi_3 \in {}^{\sigma\tau} N_l^A$, which implies ${}_{\tau^{-1}1}{}^\sigma N_v^A {}^\tau N_l^A \subseteq {}^{\sigma\tau} N_l^A$ and ${}_{\tau^{-1}3}{}^\sigma N_v^A {}^\tau N_l^A \subseteq {}^{\sigma\tau} N_l^A$.

Conversely, suppose that $\varphi = (\sigma\tau, S) \in {}^{\sigma\tau} N_l^A$, where $S = (\alpha_1, \varepsilon, \alpha_3)$ for some permutations α_1, α_3 of Q . Since ${}^\sigma N_v^A \neq \emptyset$, there exists an element say $\phi \in {}^\sigma N_v^A$. Then there exist permutations ϕ_1, ϕ_2, ϕ_3 such that $\phi = (\sigma, (\phi_1, \phi_2, \phi_3))$, where $\phi_{\tau^{-1}2} = \varepsilon$. Let $T = (\phi_1, \phi_2, \phi_3)$. As we know that $\phi, \varphi \in \text{Aus}(Q, \cdot)$ and $\text{Aus}(Q, \cdot)$ is a group, we have $\phi^{-1}\varphi \in \text{Aus}(Q, \cdot)$. Also

$$\phi^{-1}\varphi = (\sigma, T)^{-1}(\sigma\tau, S) = (\sigma^{-1}, (T^{-1})\sigma^{-1})(\sigma\tau, S) = (\tau, (T^{-1})^\tau S).$$

On substituting the values of T and S , we get

$$\begin{aligned} \phi^{-1}\varphi &= (\tau, (\phi_{\tau^{-1}1}^{-1}\alpha_1, \phi_{\tau^{-1}2}^{-1}, \phi_{\tau^{-1}3}^{-1}\alpha_3)) \\ &= (\tau, ((\phi_{\tau^{-1}1}^{-1}\alpha_1, \varepsilon, \phi_{\tau^{-1}3}^{-1}\alpha_3)) = \psi(\text{say}) \in {}^\tau N_l^A \end{aligned}$$

(because the second component is identity). Hence, for any element $\varphi \in {}^{\sigma\tau} N_l^A$, there exists an element $\phi \in {}^\sigma N_v^A$ and an element $\psi \in {}^\tau N_l^A$ such that $\phi\psi = \varphi$, i.e., ${}^{\sigma\tau} N_l^A \subseteq {}^\sigma N_v^A {}^\tau N_l^A$. Also we get for $\alpha_i \in {}^{\sigma\tau} N_l^A$ there exists $\phi_{\tau^{-1}i} \in {}_{\tau^{-1}1}{}^\sigma N_v^A$ and $\phi_{\tau^{-1}i}^{-1}\alpha_i \in {}^\tau N_l^A$ such that $(\phi_{\tau^{-1}i})(\phi_{\tau^{-1}i}^{-1}\alpha_i) = \alpha_i$, for $i = 1, 3$. Thus ${}^{\sigma\tau} N_l^A \subseteq {}_{\tau^{-1}1}{}^\sigma N_v^A {}^\tau N_l^A$ and ${}^{\sigma\tau} N_l^A \subseteq {}_{\tau^{-1}3}{}^\sigma N_v^A {}^\tau N_l^A$.

(2) and (3) can be proved on similar lines. $\square$

Table 3 is obtained using Theorem 3.2 that shows relationships between components of σ -A-nuclei, τ -A-nuclei and $\sigma\tau$ -A-nuclei of a quasigroup $(Q, \cdot)$, for all $\sigma, \tau \in S_3$.

Let quasigroup $(Q, \circ)$ be an isotrophic image of a quasigroup $(Q, *)$ with an isotrophism θ , i.e., $(Q, \circ) = (Q, *)\theta$. In order to distinguish the σ -A-nuclei of the quasigroups $(Q, *)$ and $(Q, \circ)$, we shall replace the symbol A (which stands for autotopy) by $*$ and $\circ$ (the binary operations), respectively.

The following theorem gives a complete description of left, right and middle σ -A-nuclei of isotrophic images of the form $(Q, \circ) = (Q, *)\theta$ of a quasigroup $(Q, *)$ and their respective component sets, for any $\tau \in S_3$.

Theorem 3.3. *If quasigroup $(Q, \circ)$ is an isotrophic image of a quasigroup $(Q, *)$ with an isotrophism $\theta = (\tau, (\alpha_1, \alpha_2, \alpha_3))$, i.e., $(Q, \circ) = (Q, *)\theta$, where $\tau \in S_3$, then*

the following hold:

(1) ${}^\sigma N_l^\circ = \theta^{-1} \tau \sigma \tau^{-1} N_v^* \theta \iff \sigma = \varepsilon \text{ or } (13), \text{ where}$

$$v = \begin{cases} r & \text{if } \tau^{-1}2 = 1, \\ l & \text{if } \tau^{-1}2 = 2, \\ m & \text{if } \tau^{-1}2 = 3. \end{cases}$$

Further, if $\sigma \in \{\varepsilon, (13)\}$, then ${}^\sigma_1 N_l^\circ = \alpha_{\sigma^{-1}1}^{-1} \tau \sigma \tau^{-1} N_v^* \alpha_1$ and ${}^\sigma_3 N_l^\circ = \alpha_{\sigma^{-1}3}^{-1} \tau \sigma \tau^{-1} N_v^* \alpha_3$.

(2) ${}^\sigma N_r^\circ = \theta^{-1} \tau \sigma \tau^{-1} N_v^* \theta \iff \sigma = \varepsilon \text{ or } (23), \text{ where}$

$$v = \begin{cases} r & \text{if } \tau^{-1}1 = 1, \\ l & \text{if } \tau^{-1}1 = 2, \\ m & \text{if } \tau^{-1}1 = 3. \end{cases}$$

Further, if $\sigma \in \{\varepsilon, (23)\}$ then ${}^\sigma_2 N_r^\circ = \alpha_{\sigma^{-1}2}^{-1} \tau \sigma \tau^{-1} N_v^* \alpha_2$ and ${}^\sigma_3 N_r^\circ = \alpha_{\sigma^{-1}3}^{-1} \tau \sigma \tau^{-1} N_v^* \alpha_3$.

(3) ${}^\sigma N_m^\circ = \theta^{-1} \tau \sigma \tau^{-1} N_v^* \theta \iff \sigma = \varepsilon \text{ or } (12), \text{ where}$

$$v = \begin{cases} r & \text{if } \tau^{-1}3 = 1, \\ l & \text{if } \tau^{-1}3 = 2, \\ m & \text{if } \tau^{-1}3 = 3. \end{cases}$$

Further, if $\sigma \in \{\varepsilon, (12)\}$ then ${}^\sigma_1 N_m^\circ = \alpha_{\sigma^{-1}1}^{-1} \tau \sigma \tau^{-1} N_v^* \alpha_1$ and ${}^\sigma_2 N_m^\circ = \alpha_{\sigma^{-1}2}^{-1} \tau \sigma \tau^{-1} N_v^* \alpha_2$.

Proof. (1) We shall show the inclusions in both directions. Let $T = (\alpha_1, \alpha_2, \alpha_3)$, i.e. $\theta = (\tau, T)$ and $\varphi \in {}^\sigma N_l^\circ$. Then there exist permutations $\varphi_1, \varphi_2, \varphi_3$ of Q such that $\varphi = (\sigma, (\varphi_1, \varphi_2, \varphi_3)) = (\sigma, R)$ (say) where $\varphi_2 = \varepsilon$. We have $\varphi \in \text{Aus}(Q, \circ)$ and by Theorem 2.1, $\theta \varphi \theta^{-1} \in \text{Aus}(Q, *)$. Also

$$\begin{aligned} \theta \varphi \theta^{-1} &= (\tau, T)(\sigma, R)(\tau, T)^{-1} \\ &= (\tau, T)(\sigma \tau^{-1}, R^{\tau^{-1}}(T^{-1})^{\tau^{-1}}) \\ &= (\tau \sigma \tau^{-1}, T^{\sigma \tau^{-1}} R^{\tau^{-1}}(T^{-1})^{\tau^{-1}}). \end{aligned}$$

On substituting the values of T and R we get

$$\begin{aligned} \theta \varphi \theta^{-1} &= (\tau \sigma \tau^{-1}, (\alpha_{(\tau \sigma^{-1})1} \varphi_{\tau 1} \alpha_{\tau 1}^{-1}, \alpha_{(\tau \sigma^{-1})2} \varphi_{\tau 2} \alpha_{\tau 2}^{-1}, \alpha_{(\tau \sigma^{-1})3} \varphi_{\tau 3} \alpha_{\tau 3}^{-1})) \\ &= (\tau \sigma \tau^{-1}, (\beta_1, \beta_2, \beta_3)) \text{ (say)}. \end{aligned}$$

Comparing the components we obtain $\beta_i = \alpha_{(\tau \sigma^{-1})i} \varphi_{\tau i} \alpha_{\tau i}^{-1}$, i.e. $\beta_{\tau^{-1}i} = \alpha_{\sigma^{-1}i} \varphi_i \alpha_i^{-1}$ for $i = 1, 2, 3$. As $\varphi_2 = \varepsilon$, for $i = 2$ we have $\beta_{\tau^{-1}2} = \alpha_{\sigma^{-1}2} \alpha_2^{-1}$. Thus $\theta \varphi \theta^{-1} \in$

$\tau\sigma\tau^{-1}N_v^*$ if and only if $\beta_{\tau^{-1}2} = \varepsilon$ if and only if $\alpha_{\sigma^{-1}2}\alpha_2^{-1} = \varepsilon$, i.e. $\sigma = \varepsilon$ or (13) (since $\sigma \in S_3$). Hence

$$\theta^\sigma N_l^\circ \theta^{-1} \subseteq {}^{\tau\sigma\tau^{-1}}N_v^* \iff \sigma = \varepsilon \quad \text{or} \quad (13). \quad (3.8)$$

Also, for $\sigma \in \{\varepsilon, (13)\}$ and $i = 1, 3$, we obtain $\alpha_{\sigma^{-1}i}\varphi_i\alpha_i^{-1} = \beta_{\tau^{-1}i} \in {}^{\tau\sigma\tau^{-1}}N_v^*$, that gives $\alpha_{\sigma^{-1}1} {}^\sigma N_l^\circ \alpha_1^{-1} \subseteq {}^{\tau\sigma\tau^{-1}}N_v^*$ and $\alpha_{\sigma^{-1}3} {}^\sigma N_l^\circ \alpha_3^{-1} \subseteq {}^{\tau\sigma\tau^{-1}}N_v^*$.

Conversely, let $\varphi = (\tau\sigma\tau^{-1}, R) \in {}^{\tau\sigma\tau^{-1}}N_v^*$. Then there exist permutations $\varphi_1, \varphi_2, \varphi_3$ of Q such that $R = (\varphi_1, \varphi_2, \varphi_3)$, where $\varphi_{\tau^{-1}2} = \varepsilon$. We have $\varphi \in \text{Aus}(Q, *)$, and by Theorem 2.1, we get $\theta^{-1}\varphi\theta \in \text{Aus}(Q, \circ)$. Also

$$\begin{aligned} \theta^{-1}\varphi\theta &= (\tau, T)^{-1}(\tau\sigma\tau^{-1}, R)(\tau, T) \\ &= (\tau^{-1}, (T^{-1})^{\tau^{-1}})(\tau\sigma, R^T T) \\ &= (\sigma, (T^{-1})^\sigma R^T T). \end{aligned}$$

On substituting the values of T and R , and as $\varphi_{\tau^{-1}2} = \varepsilon$, we get $\theta^{-1}\varphi\theta = (\sigma, (\alpha_{\sigma^{-1}1}\varphi_{\tau^{-1}1}\alpha_1, \alpha_{\sigma^{-1}2}\alpha_2, \alpha_{\sigma^{-1}3}\varphi_{\tau^{-1}3}\alpha_3)) \in {}^\sigma N_l^\circ$ if and only if $\alpha_{\sigma^{-1}2}\alpha_2^{-1} = \varepsilon$, i.e. $\sigma = \varepsilon$ or (13). Hence

$$\theta^{-1} {}^{\tau\sigma\tau^{-1}}N_v^* \theta \subseteq {}^\sigma N_l^\circ \iff \sigma = \varepsilon \quad \text{or} \quad (13). \quad (3.9)$$

Also, for $\sigma \in \{\varepsilon, (13)\}$, we obtain $\alpha_{\sigma^{-1}1}\varphi_{\tau^{-1}1}\alpha_1 \in {}^\sigma N_l^\circ$ and $\alpha_{\sigma^{-1}3}\varphi_{\tau^{-1}3}\alpha_3 \in {}^\sigma N_l^\circ$. Therefore $\alpha_{\sigma^{-1}1} {}^{\tau\sigma\tau^{-1}}N_v^* \alpha_1 \subseteq {}^\sigma N_l^\circ$ and $\alpha_{\sigma^{-1}3} {}^{\tau\sigma\tau^{-1}}N_v^* \alpha_3 \subseteq {}^\sigma N_l^\circ$. From (3.8) and (3.9) we obtain ${}^\sigma N_l^\circ = \theta^{-1} {}^{\tau\sigma\tau^{-1}}N_v^* \theta \iff \sigma = \varepsilon$ or (13). Also for $\sigma \in \{\varepsilon, (13)\}$, we have ${}^\sigma N_l^\circ = \alpha_{\sigma^{-1}1} {}^{\tau\sigma\tau^{-1}}N_v^* \alpha_1$ and ${}^\sigma N_l^\circ = \alpha_{\sigma^{-1}3} {}^{\tau\sigma\tau^{-1}}N_v^* \alpha_3$.

(2) and (3) can be proved on similar lines. $\square$

For $\sigma \in \{(13), (23), (12)\}$, Table 4 shows connections between components of σ -A-nuclei of a quasigroup $(Q, *)$ and components of σ -A-nuclei of its isotrophic images of the form $(Q, \circ) = (Q, *) (\tau, T)$, where $T = (\alpha, \beta, \gamma)$ and α, β, γ are permutations of the set Q , for all $\tau \in S_3$.

Note: We already have Table 1 that shows connections between components of ε -A-nuclei of a quasigroup and its isotrophic images. Hence we have omitted the case when $\sigma = \varepsilon$, as left (right, middle) ε -A-nucleus of a quasigroup coincides with the left (right, middle) A-nucleus of the quasigroup.

Corollary 3.1. *Isotrophic quasigroups have isomorphic components of σ -A-nucleus, for $\sigma \in \{\varepsilon, (12), (13), (23)\}$.*

Proof. The proof follows from Table 4. $\square$

Let $(Q, \circ)$ be a τ -parastrophe of a quasigroup $(Q, *)$, where $\tau \in S_3$. Then $(Q, \circ) = (Q, *^\tau) = (Q, *) (\tau, \varepsilon)$. Here the symbol ε denotes $(\varepsilon, \varepsilon, \varepsilon)$, where ε is the identity mapping.

Table 4. Connections between components of σ -A-nuclei of a quasigroup and its isotrophic images.

	(ε, T)	$((12), T)$	$((13), T)$	$((23), T)$	$((123), T)$	$((132), T)$
$\begin{smallmatrix} (13) \\ 1 \\ N_l^\circ \end{smallmatrix}$	$\gamma^{-1} \begin{smallmatrix} (13) \\ 1 \\ N_l^* \alpha \end{smallmatrix}$	$\gamma^{-1} \begin{smallmatrix} (23) \\ 2 \\ N_r^* \alpha \end{smallmatrix}$	$\gamma^{-1} \begin{smallmatrix} (13) \\ 3 \\ N_l^* \alpha \end{smallmatrix}$	$\gamma^{-1} \begin{smallmatrix} (12) \\ 1 \\ N_m^* \alpha \end{smallmatrix}$	$\gamma^{-1} \begin{smallmatrix} (23) \\ 3 \\ N_r^* \alpha \end{smallmatrix}$	$\gamma^{-1} \begin{smallmatrix} (12) \\ 2 \\ N_m^* \alpha \end{smallmatrix}$
$\begin{smallmatrix} (13) \\ 3 \\ N_l^\circ \end{smallmatrix}$	$\alpha^{-1} \begin{smallmatrix} (13) \\ 3 \\ N_l^* \gamma \end{smallmatrix}$	$\alpha^{-1} \begin{smallmatrix} (23) \\ 3 \\ N_r^* \gamma \end{smallmatrix}$	$\alpha^{-1} \begin{smallmatrix} (13) \\ 1 \\ N_l^* \gamma \end{smallmatrix}$	$\alpha^{-1} \begin{smallmatrix} (12) \\ 2 \\ N_m^* \gamma \end{smallmatrix}$	$\alpha^{-1} \begin{smallmatrix} (23) \\ 2 \\ N_r^* \gamma \end{smallmatrix}$	$\alpha^{-1} \begin{smallmatrix} (12) \\ 1 \\ N_m^* \gamma \end{smallmatrix}$
$\begin{smallmatrix} (23) \\ 2 \\ N_r^\circ \end{smallmatrix}$	$\gamma^{-1} \begin{smallmatrix} (23) \\ 2 \\ N_r^* \beta \end{smallmatrix}$	$\gamma^{-1} \begin{smallmatrix} (13) \\ 1 \\ N_l^* \beta \end{smallmatrix}$	$\gamma^{-1} \begin{smallmatrix} (12) \\ 2 \\ N_m^* \beta \end{smallmatrix}$	$\gamma^{-1} \begin{smallmatrix} (23) \\ 3 \\ N_r^* \beta \end{smallmatrix}$	$\gamma^{-1} \begin{smallmatrix} (12) \\ 1 \\ N_m^* \beta \end{smallmatrix}$	$\gamma^{-1} \begin{smallmatrix} (13) \\ 3 \\ N_l^* \beta \end{smallmatrix}$
$\begin{smallmatrix} (23) \\ 3 \\ N_r^\circ \end{smallmatrix}$	$\beta^{-1} \begin{smallmatrix} (23) \\ 3 \\ N_r^* \gamma \end{smallmatrix}$	$\beta^{-1} \begin{smallmatrix} (13) \\ 3 \\ N_l^* \gamma \end{smallmatrix}$	$\beta^{-1} \begin{smallmatrix} (12) \\ 1 \\ N_m^* \gamma \end{smallmatrix}$	$\beta^{-1} \begin{smallmatrix} (23) \\ 2 \\ N_r^* \gamma \end{smallmatrix}$	$\beta^{-1} \begin{smallmatrix} (12) \\ 2 \\ N_m^* \gamma \end{smallmatrix}$	$\beta^{-1} \begin{smallmatrix} (13) \\ 1 \\ N_l^* \gamma \end{smallmatrix}$
$\begin{smallmatrix} (12) \\ 1 \\ N_m^\circ \end{smallmatrix}$	$\beta^{-1} \begin{smallmatrix} (12) \\ 1 \\ N_m^* \alpha \end{smallmatrix}$	$\beta^{-1} \begin{smallmatrix} (12) \\ 2 \\ N_m^* \alpha \end{smallmatrix}$	$\beta^{-1} \begin{smallmatrix} (23) \\ 3 \\ N_r^* \alpha \end{smallmatrix}$	$\beta^{-1} \begin{smallmatrix} (13) \\ 1 \\ N_l^* \alpha \end{smallmatrix}$	$\beta^{-1} \begin{smallmatrix} (13) \\ 3 \\ N_l^* \alpha \end{smallmatrix}$	$\beta^{-1} \begin{smallmatrix} (23) \\ 2 \\ N_r^* \alpha \end{smallmatrix}$
$\begin{smallmatrix} (12) \\ 2 \\ N_m^\circ \end{smallmatrix}$	$\alpha^{-1} \begin{smallmatrix} (12) \\ 2 \\ N_m^* \beta \end{smallmatrix}$	$\alpha^{-1} \begin{smallmatrix} (12) \\ 1 \\ N_m^* \beta \end{smallmatrix}$	$\alpha^{-1} \begin{smallmatrix} (23) \\ 2 \\ N_r^* \beta \end{smallmatrix}$	$\alpha^{-1} \begin{smallmatrix} (13) \\ 3 \\ N_l^* \beta \end{smallmatrix}$	$\alpha^{-1} \begin{smallmatrix} (13) \\ 1 \\ N_l^* \beta \end{smallmatrix}$	$\alpha^{-1} \begin{smallmatrix} (23) \\ 3 \\ N_r^* \beta \end{smallmatrix}$

The following theorem gives a complete description of left, right and middle σ -A-nuclei of τ -parastrophes $(Q, \circ)$ of a quasigroup $(Q, *)$ and their respective component sets, for any $\sigma, \tau \in S_3$.

Theorem 3.4. *Let $(Q, *)$ be a quasigroup and $(Q, \circ)$ be a τ -parastrophe of $(Q, *)$, i.e., $\circ = *^\tau$, where $\tau \in S_3$. If $\theta = (\tau, \varepsilon)$, then for all $\sigma \in S_3$ the following hold:*

(1) ${}^\sigma N_l^\circ = \theta^{-1} {}^{\tau\sigma\tau^{-1}} N_v^* \theta$, where

$$v = \begin{cases} r & \text{if } \tau^{-1}2 = 1, \\ l & \text{if } \tau^{-1}2 = 2, \\ m & \text{if } \tau^{-1}2 = 3. \end{cases}$$

Further, ${}^\sigma N_l^\circ = {}^{\tau\sigma\tau^{-1}} N_v^*$ and ${}^\sigma N_l^\circ = {}^{\tau\sigma\tau^{-1}} N_v^*$.

(2) ${}^\sigma N_r^\circ = \theta^{-1} {}^{\tau\sigma\tau^{-1}} N_v^* \theta$, where

$$v = \begin{cases} r & \text{if } \tau^{-1}1 = 1, \\ l & \text{if } \tau^{-1}1 = 2, \\ m & \text{if } \tau^{-1}1 = 3. \end{cases}$$

Further, ${}^\sigma N_r^\circ = {}^{\tau\sigma\tau^{-1}} N_v^*$ and ${}^\sigma N_r^\circ = {}^{\tau\sigma\tau^{-1}} N_v^*$.

(3) ${}^\sigma N_m^\circ = \theta^{-1} {}^{\tau\sigma\tau^{-1}} N_v^* \theta$, where

$$v = \begin{cases} r & \text{if } \tau^{-1}3 = 1, \\ l & \text{if } \tau^{-1}3 = 2, \\ m & \text{if } \tau^{-1}3 = 3. \end{cases}$$

Further, ${}^\sigma N_m^\circ = {}^{\tau\sigma\tau^{-1}} N_v^*$ and ${}^\sigma N_m^\circ = {}^{\tau\sigma\tau^{-1}} N_v^*$.

Proof. (1) We shall show the inclusions in both directions. Let $\varphi \in {}^\sigma N_l^\circ$. Then there exist permutations $\varphi_1, \varphi_2, \varphi_3$ of Q such that $\varphi = (\sigma, (\varphi_1, \varphi_2, \varphi_3)) = (\sigma, R)$ (say), where $\varphi_2 = \varepsilon$. Therefore $\varphi \in \text{Aus}(Q, \circ)$, and by Theorem 2.1, we have

$\theta\varphi\theta^{-1} \in \text{Aus}(Q, *)$. Also

$$\begin{aligned}\theta\varphi\theta^{-1} &= (\tau, \epsilon)(\sigma, R)(\tau, \epsilon)^{-1} \\ &= (\tau\sigma\tau^{-1}, R^{\tau^{-1}}) \\ &= (\tau\sigma\tau^{-1}, (\varphi_{\tau 1}, \varphi_{\tau 2}, \varphi_{\tau 3})) \\ &= (\tau\sigma\tau^{-1}, (\alpha_1, \alpha_2, \alpha_3)) \text{ (say).}\end{aligned}$$

Table 5. Connections between components of σ -A-nuclei of a quasigroup and its τ -parastrophes.

$\sigma \backslash \tau$		ϵ	(12)	(13)	(23)	(123)	(132)
ϵ	$\overset{\epsilon}{1}N^\circ_l$	$\overset{\epsilon}{1}N^*_l$	$\overset{\epsilon}{2}N^*_r$	$\overset{\epsilon}{3}N^*_l$	$\overset{\epsilon}{1}N^*_m$	$\overset{\epsilon}{3}N^*_r$	$\overset{\epsilon}{2}N^*_m$
	$\overset{\epsilon}{3}N^\circ_l$	$\overset{\epsilon}{3}N^*_l$	$\overset{\epsilon}{3}N^*_r$	$\overset{\epsilon}{1}N^*_l$	$\overset{\epsilon}{2}N^*_m$	$\overset{\epsilon}{2}N^*_r$	$\overset{\epsilon}{1}N^*_m$
	$\overset{\epsilon}{2}N^\circ_r$	$\overset{\epsilon}{2}N^*_r$	$\overset{\epsilon}{1}N^*_l$	$\overset{\epsilon}{2}N^*_m$	$\overset{\epsilon}{3}N^*_r$	$\overset{\epsilon}{1}N^*_m$	$\overset{\epsilon}{3}N^*_l$
	$\overset{\epsilon}{3}N^\circ_r$	$\overset{\epsilon}{3}N^*_r$	$\overset{\epsilon}{3}N^*_l$	$\overset{\epsilon}{1}N^*_m$	$\overset{\epsilon}{2}N^*_r$	$\overset{\epsilon}{2}N^*_m$	$\overset{\epsilon}{1}N^*_l$
	$\overset{\epsilon}{1}N^\circ_m$	$\overset{\epsilon}{1}N^*_m$	$\overset{\epsilon}{2}N^*_m$	$\overset{\epsilon}{3}N^*_r$	$\overset{\epsilon}{1}N^*_l$	$\overset{\epsilon}{3}N^*_l$	$\overset{\epsilon}{2}N^*_r$
	$\overset{\epsilon}{2}N^\circ_m$	$\overset{\epsilon}{2}N^*_m$	$\overset{\epsilon}{1}N^*_m$	$\overset{\epsilon}{2}N^*_r$	$\overset{\epsilon}{3}N^*_l$	$\overset{\epsilon}{1}N^*_l$	$\overset{\epsilon}{3}N^*_r$
	$\overset{\epsilon}{3}N^\circ_m$	$\overset{\epsilon}{3}N^*_m$	$\overset{\epsilon}{3}N^*_l$	$\overset{\epsilon}{1}N^*_r$	$\overset{\epsilon}{2}N^*_m$	$\overset{\epsilon}{2}N^*_r$	$\overset{\epsilon}{1}N^*_m$
(12)	$\overset{(12)}{1}N^\circ_l$	$\overset{(12)}{1}N^*_l$	$\overset{(12)}{2}N^*_r$	$\overset{(23)}{3}N^*_l$	$\overset{(13)}{1}N^*_m$	$\overset{(13)}{3}N^*_r$	$\overset{(23)}{2}N^*_m$
	$\overset{(12)}{3}N^\circ_l$	$\overset{(12)}{3}N^*_l$	$\overset{(12)}{3}N^*_r$	$\overset{(23)}{1}N^*_l$	$\overset{(13)}{2}N^*_m$	$\overset{(13)}{2}N^*_r$	$\overset{(23)}{1}N^*_m$
	$\overset{(12)}{2}N^\circ_r$	$\overset{(12)}{2}N^*_r$	$\overset{(12)}{1}N^*_l$	$\overset{(23)}{2}N^*_m$	$\overset{(13)}{3}N^*_r$	$\overset{(13)}{1}N^*_m$	$\overset{(23)}{3}N^*_l$
	$\overset{(12)}{3}N^\circ_r$	$\overset{(12)}{3}N^*_r$	$\overset{(12)}{3}N^*_l$	$\overset{(23)}{1}N^*_m$	$\overset{(13)}{2}N^*_r$	$\overset{(13)}{2}N^*_m$	$\overset{(23)}{1}N^*_l$
	$\overset{(12)}{1}N^\circ_m$	$\overset{(12)}{1}N^*_m$	$\overset{(12)}{2}N^*_m$	$\overset{(23)}{3}N^*_r$	$\overset{(13)}{1}N^*_l$	$\overset{(13)}{3}N^*_l$	$\overset{(23)}{2}N^*_r$
	$\overset{(12)}{2}N^\circ_m$	$\overset{(12)}{2}N^*_m$	$\overset{(12)}{1}N^*_m$	$\overset{(23)}{2}N^*_r$	$\overset{(13)}{3}N^*_l$	$\overset{(13)}{2}N^*_l$	$\overset{(23)}{3}N^*_r$
	$\overset{(12)}{3}N^\circ_m$	$\overset{(12)}{3}N^*_m$	$\overset{(12)}{3}N^*_l$	$\overset{(23)}{1}N^*_r$	$\overset{(13)}{2}N^*_m$	$\overset{(13)}{2}N^*_r$	$\overset{(23)}{1}N^*_m$
(13)	$\overset{(13)}{1}N^\circ_l$	$\overset{(13)}{1}N^*_l$	$\overset{(23)}{2}N^*_r$	$\overset{(13)}{3}N^*_l$	$\overset{(12)}{1}N^*_m$	$\overset{(23)}{3}N^*_r$	$\overset{(12)}{2}N^*_m$
	$\overset{(13)}{3}N^\circ_l$	$\overset{(13)}{3}N^*_l$	$\overset{(23)}{3}N^*_r$	$\overset{(13)}{1}N^*_l$	$\overset{(12)}{2}N^*_m$	$\overset{(23)}{2}N^*_r$	$\overset{(12)}{1}N^*_m$
	$\overset{(13)}{2}N^\circ_r$	$\overset{(13)}{2}N^*_r$	$\overset{(23)}{1}N^*_l$	$\overset{(13)}{2}N^*_m$	$\overset{(12)}{3}N^*_r$	$\overset{(23)}{1}N^*_m$	$\overset{(12)}{3}N^*_l$
	$\overset{(13)}{3}N^\circ_r$	$\overset{(13)}{3}N^*_r$	$\overset{(23)}{3}N^*_l$	$\overset{(13)}{1}N^*_m$	$\overset{(12)}{2}N^*_r$	$\overset{(23)}{2}N^*_m$	$\overset{(12)}{1}N^*_l$
	$\overset{(13)}{1}N^\circ_m$	$\overset{(13)}{1}N^*_m$	$\overset{(23)}{2}N^*_m$	$\overset{(13)}{3}N^*_r$	$\overset{(12)}{1}N^*_l$	$\overset{(23)}{3}N^*_l$	$\overset{(12)}{2}N^*_r$
	$\overset{(13)}{2}N^\circ_m$	$\overset{(13)}{2}N^*_m$	$\overset{(23)}{1}N^*_m$	$\overset{(13)}{2}N^*_r$	$\overset{(12)}{3}N^*_l$	$\overset{(23)}{2}N^*_l$	$\overset{(12)}{3}N^*_r$
	$\overset{(13)}{3}N^\circ_m$	$\overset{(13)}{3}N^*_m$	$\overset{(23)}{3}N^*_l$	$\overset{(13)}{1}N^*_r$	$\overset{(12)}{2}N^*_m$	$\overset{(23)}{1}N^*_r$	$\overset{(12)}{2}N^*_m$
(23)	$\overset{(23)}{1}N^\circ_l$	$\overset{(23)}{1}N^*_l$	$\overset{(13)}{2}N^*_r$	$\overset{(12)}{3}N^*_l$	$\overset{(23)}{2}N^*_m$	$\overset{(12)}{3}N^*_r$	$\overset{(13)}{2}N^*_m$
	$\overset{(23)}{3}N^\circ_l$	$\overset{(23)}{3}N^*_l$	$\overset{(13)}{3}N^*_r$	$\overset{(12)}{1}N^*_l$	$\overset{(23)}{3}N^*_m$	$\overset{(12)}{2}N^*_r$	$\overset{(13)}{1}N^*_m$
	$\overset{(23)}{2}N^\circ_r$	$\overset{(23)}{2}N^*_r$	$\overset{(13)}{1}N^*_l$	$\overset{(12)}{2}N^*_m$	$\overset{(23)}{3}N^*_r$	$\overset{(12)}{1}N^*_m$	$\overset{(13)}{3}N^*_l$
	$\overset{(23)}{3}N^\circ_r$	$\overset{(23)}{3}N^*_r$	$\overset{(13)}{3}N^*_l$	$\overset{(12)}{1}N^*_m$	$\overset{(23)}{2}N^*_r$	$\overset{(12)}{2}N^*_m$	$\overset{(13)}{1}N^*_l$
	$\overset{(23)}{1}N^\circ_m$	$\overset{(23)}{1}N^*_m$	$\overset{(13)}{2}N^*_m$	$\overset{(12)}{3}N^*_r$	$\overset{(23)}{1}N^*_l$	$\overset{(12)}{3}N^*_l$	$\overset{(13)}{2}N^*_r$
	$\overset{(23)}{2}N^\circ_m$	$\overset{(23)}{2}N^*_m$	$\overset{(13)}{3}N^*_m$	$\overset{(12)}{2}N^*_r$	$\overset{(23)}{3}N^*_l$	$\overset{(12)}{2}N^*_r$	$\overset{(13)}{3}N^*_l$
	$\overset{(23)}{3}N^\circ_m$	$\overset{(23)}{3}N^*_m$	$\overset{(13)}{1}N^*_m$	$\overset{(12)}{2}N^*_r$	$\overset{(23)}{2}N^*_l$	$\overset{(12)}{1}N^*_l$	$\overset{(13)}{3}N^*_r$
(123)	$\overset{(123)}{1}N^\circ_l$	$\overset{(123)}{1}N^*_l$	$\overset{(132)}{2}N^*_r$	$\overset{(132)}{3}N^*_l$	$\overset{(132)}{1}N^*_m$	$\overset{(123)}{3}N^*_r$	$\overset{(123)}{2}N^*_m$
	$\overset{(123)}{3}N^\circ_l$	$\overset{(123)}{3}N^*_l$	$\overset{(132)}{3}N^*_r$	$\overset{(132)}{1}N^*_l$	$\overset{(132)}{2}N^*_m$	$\overset{(123)}{2}N^*_r$	$\overset{(123)}{1}N^*_m$
	$\overset{(123)}{2}N^\circ_r$	$\overset{(123)}{2}N^*_r$	$\overset{(132)}{1}N^*_l$	$\overset{(132)}{2}N^*_m$	$\overset{(132)}{3}N^*_r$	$\overset{(123)}{1}N^*_m$	$\overset{(123)}{3}N^*_l$
	$\overset{(123)}{3}N^\circ_r$	$\overset{(123)}{3}N^*_r$	$\overset{(132)}{3}N^*_l$	$\overset{(132)}{1}N^*_m$	$\overset{(132)}{2}N^*_r$	$\overset{(123)}{2}N^*_m$	$\overset{(123)}{1}N^*_l$
	$\overset{(123)}{1}N^\circ_m$	$\overset{(123)}{1}N^*_m$	$\overset{(132)}{2}N^*_m$	$\overset{(132)}{3}N^*_r$	$\overset{(132)}{1}N^*_l$	$\overset{(123)}{3}N^*_l$	$\overset{(123)}{2}N^*_r$
	$\overset{(123)}{2}N^\circ_m$	$\overset{(123)}{2}N^*_m$	$\overset{(132)}{1}N^*_m$	$\overset{(132)}{2}N^*_r$	$\overset{(132)}{3}N^*_l$	$\overset{(123)}{1}N^*_l$	$\overset{(123)}{3}N^*_r$
	$\overset{(123)}{3}N^\circ_m$	$\overset{(123)}{3}N^*_m$	$\overset{(132)}{3}N^*_l$	$\overset{(132)}{1}N^*_r$	$\overset{(132)}{2}N^*_m$	$\overset{(123)}{2}N^*_r$	$\overset{(123)}{1}N^*_m$
(132)	$\overset{(132)}{1}N^\circ_l$	$\overset{(132)}{1}N^*_l$	$\overset{(123)}{2}N^*_r$	$\overset{(123)}{3}N^*_l$	$\overset{(123)}{1}N^*_m$	$\overset{(132)}{3}N^*_r$	$\overset{(132)}{2}N^*_m$
	$\overset{(132)}{3}N^\circ_l$	$\overset{(132)}{3}N^*_l$	$\overset{(123)}{3}N^*_r$	$\overset{(123)}{1}N^*_l$	$\overset{(123)}{2}N^*_m$	$\overset{(132)}{2}N^*_r$	$\overset{(132)}{1}N^*_m$
	$\overset{(132)}{2}N^\circ_r$	$\overset{(132)}{2}N^*_r$	$\overset{(123)}{1}N^*_l$	$\overset{(123)}{2}N^*_m$	$\overset{(123)}{3}N^*_r$	$\overset{(132)}{1}N^*_m$	$\overset{(132)}{3}N^*_l$
	$\overset{(132)}{3}N^\circ_r$	$\overset{(132)}{3}N^*_r$	$\overset{(123)}{3}N^*_l$	$\overset{(123)}{1}N^*_m$	$\overset{(123)}{2}N^*_r$	$\overset{(132)}{2}N^*_m$	$\overset{(132)}{1}N^*_l$
	$\overset{(132)}{1}N^\circ_m$	$\overset{(132)}{1}N^*_m$	$\overset{(123)}{2}N^*_m$	$\overset{(123)}{3}N^*_r$	$\overset{(123)}{1}N^*_l$	$\overset{(132)}{3}N^*_l$	$\overset{(132)}{2}N^*_r$
	$\overset{(132)}{2}N^\circ_m$	$\overset{(132)}{2}N^*_m$	$\overset{(123)}{1}N^*_m$	$\overset{(123)}{2}N^*_r$	$\overset{(123)}{3}N^*_l$	$\overset{(132)}{1}N^*_l$	$\overset{(132)}{3}N^*_r$
	$\overset{(132)}{3}N^\circ_m$	$\overset{(132)}{3}N^*_m$	$\overset{(123)}{3}N^*_l$	$\overset{(123)}{1}N^*_r$	$\overset{(123)}{2}N^*_m$	$\overset{(132)}{2}N^*_r$	$\overset{(132)}{1}N^*_m$

Then on comparing the components, we get $\alpha_i = \varphi_{\tau i}$, i.e., $\alpha_{\tau^{-1}i} = \varphi_i$ for $i = 1, 2, 3$. For $i = 2$, $\alpha_{\tau^{-1}2} = \varphi_2 = \varepsilon$, which implies $\theta\varphi\theta^{-1} \in {}^{\tau\sigma\tau^{-1}}N_v^*$. Hence $\theta {}^\sigma N_l^\circ \theta^{-1} \subseteq {}^{\tau\sigma\tau^{-1}}N_v^*$. Also we get $\varphi_i = \alpha_{\tau^{-1}i} \in {}^{\tau\sigma\tau^{-1}}N_v^*$, for $i = 1, 2, 3$. Thus ${}^\sigma N_l^\circ \subseteq {}^{\tau\sigma\tau^{-1}}N_v^*$ and ${}^\sigma N_l^\circ \subseteq {}^{\tau\sigma\tau^{-1}}N_v^*$.

Conversely, let $\varphi \in {}^{\tau\sigma\tau^{-1}}N_v^*$. Then there exist permutations $\varphi_1, \varphi_2, \varphi_3$ of Q such that $\varphi = (\tau\sigma\tau^{-1}, (\varphi_1, \varphi_2, \varphi_3)) = (\tau\sigma\tau^{-1}, R)$ (say), where $\varphi_{\tau^{-1}2} = \varepsilon$. We have $\varphi \in \text{Aus}(Q, *)$ and by Theorem 2.1 $\theta^{-1}\varphi\theta \in \text{Aus}(Q, \circ)$. Also

$$\theta^{-1}\varphi\theta = (\tau, \varepsilon)^{-1}(\tau\sigma\tau^{-1}, R)(\tau, \varepsilon) = (\sigma, R^\tau).$$

On substituting the value of R , and as $\varphi_{\tau^{-1}2} = \varepsilon$, we get $\theta^{-1}\varphi\theta = (\sigma, (\varphi_{\tau^{-1}1}, \varepsilon, \varphi_{\tau^{-1}3})) \in {}^\sigma N_l^\circ$. Hence $\theta^{-1} {}^{\tau\sigma\tau^{-1}}N_v^* \theta \subseteq {}^\sigma N_l^\circ$. Also observe that $\varphi_{\tau^{-1}1} \in {}^\sigma N_l^\circ$ and $\varphi_{\tau^{-1}3} \in {}^\sigma N_l^\circ$, thus ${}^{\tau\sigma\tau^{-1}}N_v^* \subseteq {}^\sigma N_l^\circ$ and ${}^{\tau\sigma\tau^{-1}}N_v^* \subseteq {}^\sigma N_l^\circ$.

(2) and (3) can be proved on similar lines. $\square$

Table 5 shows connections between components of σ -A-nuclei of a quasigroup $(Q, *)$ and components of σ -A-nuclei of its τ -parastrophes $(Q, \circ) = (Q, *^\tau)$, for all $\sigma, \tau \in S_3$.

Note that the τ -parastrophes $(Q, \circ) = (Q, *^\tau) = (Q, *) (\tau, \varepsilon)$ can also be considered as isotrophic images of quasigroup $(Q, *)$ with isotrophisms (τ, ε) .

4. σ -A-Nuclei in Inverse Quasigroups

In this section we will study properties of the σ -A-nuclei of various inverse quasigroups using the connections derived in Sec. 3. In Sec. 4.1, we shall discuss on (α, β, γ) -inverse quasigroups and in Sec. 4.2, we shall discuss on λ, ρ and μ -inverse quasigroups.

4.1. (α, β, γ) -inverse quasigroups

Let $(Q, \cdot)$ be a quasigroup. Then $(Q, \cdot)$ is an (α, β, γ) -inverse quasigroup if there exist permutations α, β, γ of the set Q such that

$$\alpha(x \cdot y) \cdot \beta x = \gamma y \quad (4.1)$$

for all $x, y \in Q$ [11, 20]. From definition of autostrophism, (4.1) is true if and only if $\theta = ((123), (\alpha, \beta, \gamma))$ is an autostrophism of $(Q, \cdot)$. Thus from Theorem 3.3 we get in an (α, β, γ) -inverse quasigroup ${}^{(13)}N_l^A = \theta^{-1} {}^{(23)}N_r^A \theta$, ${}^{(23)}N_r^A = \theta^{-1} {}^{(12)}N_m^A \theta$ and ${}^{(12)}N_m^A = \theta^{-1} {}^{(13)}N_l^A \theta$, which implies that the left (13)-A-nucleus is isomorphic to the right (23)-A-nucleus, the right (23)-A-nucleus is isomorphic to the middle (12)-A-nucleus and the middle (12)-A-nucleus is isomorphic to the left (13)-A-nucleus. Hence ${}^{(13)}N_l^A$, ${}^{(23)}N_r^A$ and ${}^{(12)}N_m^A$ are isomorphic in an (α, β, γ) -inverse quasigroup $(Q, \cdot)$.

Also observe from above that

$${}^{(13)}N_l^A = \theta^{-1} {}^{(23)}N_r^A \theta = (\theta^2)^{-1} {}^{(12)}N_m^A \theta^2 = (\theta^3)^{-1} {}^{(13)}N_l^A \theta^3,$$

which implies $\theta^3 \in N({}^{(13)}N_l^A)$. Similarly we have $\theta^3 \in N({}^{(23)}N_r^A)$ and $\theta^3 \in N({}^{(12)}N_m^A)$. It can easily be seen that $\theta^3 = (\beta\gamma\alpha, \gamma\alpha\beta, \alpha\beta\gamma)$. Thus in an (α, β, γ) -inverse quasigroup, we have $(\beta\gamma\alpha, \gamma\alpha\beta, \alpha\beta\gamma) \in N({}^{(13)}N_l^A) \cap N({}^{(23)}N_r^A) \cap N({}^{(12)}N_m^A)$.

A quasigroup $(Q, \cdot)$ has the *weak-inverse-property* (WIP) if there exists a permutation J of the set Q such that

$$x \cdot J(y \cdot x) = Jy \quad (4.2)$$

for all $x, y \in Q$ [2, 12]. A WIP quasigroup is a (J, ε, J) -inverse quasigroup [2], which implies that if $(Q, \cdot)$ is a WIP-quasigroup with respect to the permutation J , then $(J^2, J^2, J^2) \in N({}^{(13)}N_l^A) \cap N({}^{(23)}N_r^A) \cap N({}^{(12)}N_m^A)$.

A quasigroup $(Q, \cdot)$ has the *crossed-inverse-property* (CI) if there exists a permutation J of the set Q such that

$$(x \cdot y) \cdot Jx = y \quad (4.3)$$

for all $x, y \in Q$ [9, 12]. Thus a CI quasigroup is an $(\varepsilon, J, \varepsilon)$ -inverse quasigroup, which implies that if $(Q, \cdot)$ is a CI-quasigroup with respect to the permutation J , then $(J, J, J) \in N({}^{(13)}N_l^A) \cap N({}^{(23)}N_r^A) \cap N({}^{(12)}N_m^A)$.

A quasigroup $(Q, \cdot)$ has the (r, s, t) -inverse-property if there exists a permutation J of the set Q such that

$$J^r(x \cdot y) \cdot J^s x = J^t y \quad (4.4)$$

for all $x, y \in Q$ [11, 13]. Thus an (r, s, t) -inverse quasigroup is a (J^r, J^s, J^t) -inverse quasigroup, which implies that if $(Q, \cdot)$ is an (r, s, t) -quasigroup with respect to the permutation J then $(J^{r+s+t}, J^{r+s+t}, J^{r+s+t}) \in N({}^{(13)}N_l^A) \cap N({}^{(23)}N_r^A) \cap N({}^{(12)}N_m^A)$.

A quasigroup $(Q, \cdot)$ has the m -inverse-property if there exists a permutation J of the set Q such that

$$J^m(x \cdot y) \cdot J^{m+1} x = J^m y \quad (4.5)$$

for all $x, y \in Q$ [10–12]. Thus an m -inverse-quasigroup is a (J^m, J^{m+1}, J^m) -inverse quasigroup, which implies that if $(Q, \cdot)$ is an m -inverse-quasigroup with respect to the permutation J then $(J^{3m+1}, J^{3m+1}, J^{3m+1}) \in N({}^{(13)}N_l^A) \cap N({}^{(23)}N_r^A) \cap N({}^{(12)}N_m^A)$.

Since an (α, β, γ) -inverse quasigroup has $((123), (\alpha, \beta, \gamma))$ autostrophism, from Table 4 we conclude that in an (α, β, γ) -inverse quasigroup, we have ${}^{(13)}_1N_l^A = \gamma^{-1} {}^{(23)}_3N_r^A \alpha$, ${}^{(13)}_3N_l^A = \alpha^{-1} {}^{(23)}_2N_r^A \gamma$, ${}^{(23)}_2N_r^A = \gamma^{-1} {}^{(12)}_1N_m^A \beta$, ${}^{(23)}_3N_r^A = \beta^{-1} {}^{(12)}_2N_m^A \gamma$, ${}^{(12)}_1N_m^A = \beta^{-1} {}^{(13)}_3N_l^A \alpha$, and ${}^{(12)}_2N_m^A = \alpha^{-1} {}^{(13)}_1N_l^A \beta$. Therefore, in an $(\varepsilon, \beta, \gamma)$ -inverse quasigroup, we have ${}^{(13)}_1N_l^A = \gamma^{-1} {}^{(23)}_3N_r^A = \gamma^{-1} \beta^{-1} {}^{(12)}_2N_m^A \gamma = \gamma^{-1} \beta^{-1} {}^{(13)}_1N_l^A \beta \gamma$, which implies $\beta\gamma \in N({}^{(13)}_1N_l^A)$. Also, we have ${}^{(13)}_3N_l^A = {}^{(23)}_2N_r^A \gamma = \gamma^{-1} {}^{(12)}_1N_m^A \beta \gamma = \gamma^{-1} \beta^{-1} {}^{(13)}_3N_l^A \beta \gamma$, and hence $\beta\gamma \in N({}^{(13)}_3N_l^A)$. Thus $\beta\gamma \in N({}^{(13)}_1N_l^A) \cap N({}^{(13)}_3N_l^A)$.

Similarly, in an $(\alpha, \varepsilon, \gamma)$ -inverse quasigroup, we have $\gamma\alpha \in N(\binom{12}{1}N_m^A) \cap N(\binom{12}{2}N_m^A)$, and in an $(\alpha, \beta, \varepsilon)$ -inverse quasigroup, we have $\alpha\beta \in N(\binom{23}{2}N_r^A) \cap N(\binom{23}{3}N_r^A)$.

It can be observed that if $(Q, \cdot)$ is a WIP-quasigroup with respect to the permutation J , then $J^2 \in N(\binom{12}{1}N_m^A) \cap N(\binom{12}{2}N_m^A)$ and if $(Q, \cdot)$ is a CI-quasigroup with respect to the permutation J , then $J \in N(\binom{13}{1}N_l^A) \cap N(\binom{13}{3}N_l^A)$ and $J \in N(\binom{23}{2}N_r^A) \cap N(\binom{23}{3}N_r^A)$.

4.2. λ, ρ and μ -inverse quasigroups

A quasigroup $(Q, \cdot)$ is a λ -inverse quasigroup if there exist permutations $\lambda_1, \lambda_2, \lambda_3$ of the set Q such that

$$\lambda_1 x \cdot \lambda_2(x \cdot y) = \lambda_3 y \quad (4.6)$$

for all $x, y \in Q$ [5, 11]. From definition of autostrophism, (4.6) holds if and only if $\theta = ((23), (\lambda_1, \lambda_2, \lambda_3))$ is an autostrophism of $(Q, \cdot)$.

From Theorem 3.3, we conclude that in a λ -inverse quasigroup, we have $\binom{13}{1}N_l^A = \theta^{-1} \binom{12}{1}N_m^A \theta$, $\binom{23}{2}N_r^A = \theta^{-1} \binom{23}{2}N_r^A \theta$ and $\binom{12}{2}N_m^A = \theta^{-1} \binom{13}{1}N_l^A \theta$, which implies left (13)- A -nucleus is isomorphic to middle (12)- A -nucleus. Also, $\binom{13}{1}N_l^A = \theta^{-1} \binom{12}{1}N_m^A \theta = (\theta^2)^{-1} \binom{13}{1}N_l^A \theta^2$, which gives $\theta^2 \in N(\binom{13}{1}N_l^A)$, $\binom{23}{2}N_r^A = \theta^{-1} \binom{23}{2}N_r^A \theta$ gives $\theta \in N(\binom{23}{2}N_r^A)$ and $\binom{12}{2}N_m^A = \theta^{-1} \binom{13}{1}N_l^A \theta = (\theta^2)^{-1} \binom{12}{2}N_m^A \theta^2$ implies $\theta^2 \in N(\binom{12}{2}N_m^A)$. Thus in a λ -inverse quasigroup, $\theta \in N(\binom{23}{2}N_r^A)$ and $\theta^2 \in N(\binom{13}{1}N_l^A) \cap N(\binom{12}{2}N_m^A)$.

From Table 4, we have in a λ -inverse quasigroup $\binom{13}{1}N_l^A = \lambda_3^{-1} \binom{12}{1}N_m^A \lambda_1$, $\binom{13}{3}N_r^A = \lambda_1^{-1} \binom{12}{2}N_m^A \lambda_3$, $\binom{23}{2}N_r^A = \lambda_3^{-1} \binom{23}{2}N_r^A \lambda_2$, $\binom{23}{3}N_r^A = \lambda_2^{-1} \binom{23}{3}N_r^A \lambda_3$, $\binom{12}{1}N_m^A = \lambda_2^{-1} \binom{13}{1}N_l^A \lambda_1$, and $\binom{12}{2}N_m^A = \lambda_1^{-1} \binom{13}{3}N_r^A \lambda_2$.

A quasigroup $(Q, \cdot)$ has the *left-inverse-property (LIP)* with respect to a permutation λ of Q if

$$\lambda x \cdot xy = y \quad (4.7)$$

for all $x, y \in Q$ [2, 20]. Note that (4.7) is true if and only if $((23), (\lambda, \varepsilon, \varepsilon))$ is an autostrophism of $(Q, \cdot)$. Therefore, an LIP-quasigroup is a λ -inverse quasigroup with $\lambda_1 = \lambda$ and $\lambda_2 = \lambda_3 = \varepsilon$. Also from (4.7) we have $\lambda^2 x (\lambda x \cdot xy) = xy$, i.e. $\lambda^2 x \cdot y = x \cdot y$, which gives $\lambda^2 = \varepsilon$. Hence in LIP-quasigroup $\binom{13}{1}N_l^A = \binom{12}{1}N_m^A \lambda$, $\binom{13}{3}N_r^A = \lambda \binom{12}{2}N_m^A$ and $\binom{23}{2}N_r^A = \binom{23}{3}N_r^A$.

A quasigroup $(Q, \cdot)$ is a ρ -inverse quasigroup if there exist permutations ρ_1, ρ_2, ρ_3 of the set Q such that

$$\rho_1(x \cdot y) \cdot \rho_2 y = \rho_3 x \quad (4.8)$$

for all $x, y \in Q$ [5, 11]. From definition of autostrophism, (4.8) is true if and only if $\theta = ((13), (\rho_1, \rho_2, \rho_3))$ is an autostrophism of $(Q, \cdot)$.

From Theorem 3.3, we conclude in a ρ -inverse quasigroup, we have $\binom{13}{1}N_l^A = \theta^{-1} \binom{13}{1}N_l^A \theta$, $\binom{23}{2}N_r^A = \theta^{-1} \binom{12}{2}N_m^A \theta$ and $\binom{12}{2}N_m^A = \theta^{-1} \binom{23}{2}N_r^A \theta$, which implies

right (23)-A-nucleus is isomorphic to middle (12)-A-nucleus. Also observe that in a ρ -inverse quasigroup $\theta \in N({}^{(13)}N_l^A)$ and $\theta^2 \in N({}^{(23)}N_r^A) \cap N({}^{(12)}N_m^A)$.

From Table 4 we conclude in a ρ -inverse quasigroup, we have ${}^{(13)}N_l^A = \rho_3^{-1}({}^{(13)}N_l^A \rho_1, {}^{(13)}N_l^A = \rho_1^{-1}({}^{(13)}N_l^A \rho_3, {}^{(23)}N_r^A = \rho_3^{-1}({}^{(23)}N_r^A \rho_2, {}^{(23)}N_r^A = \rho_2^{-1}({}^{(23)}N_r^A \rho_3, {}^{(12)}N_m^A = \rho_2^{-1}({}^{(12)}N_m^A \rho_1$ and ${}^{(12)}N_m^A = \rho_1^{-1}({}^{(12)}N_m^A \rho_2$.

A quasigroup $(Q, \cdot)$ has the *right inverse property (RIP)* with respect to a permutation ρ of Q if

$$xy \cdot \rho y = x \quad (4.9)$$

for all $x, y \in Q$ [2, 20]. Note that (4.9) holds if and only if $((13), (\varepsilon, \rho, \varepsilon))$ is an autostrophism of $(Q, \cdot)$. Therefore, an LIP-quasigroup is a ρ -inverse quasigroup with $\rho_2 = \rho$ and $\rho_1 = \rho_3 = \varepsilon$. Also from (4.9), $(xy \cdot \rho y) \rho^2 y = xy$, i.e., $x \cdot \rho^2 y = x \cdot y$ which implies $\rho^2 = \varepsilon$. Hence, in an RIP-quasigroup ${}^{(13)}N_l^A = {}^{(13)}N_l^A, {}^{(23)}N_r^A = {}^{(12)}N_m^A \rho$, and ${}^{(12)}N_m^A = \rho({}^{(23)}N_r^A)$.

A quasigroup $(Q, \cdot)$ has the *inverse property (IP)* (with respect to permutations λ and ρ of Q) if it is both LIP and RIP-inverse quasigroup [2]. Thus $\lambda^2 = \varepsilon$ and $\rho^2 = \varepsilon$. Hence in IP-quasigroup ${}^{(13)}N_l^A = {}^{(13)}N_l^A, {}^{(23)}N_r^A = {}^{(23)}N_r^A$, and ${}^{(12)}N_m^A \cong {}^{(12)}N_m^A$.

A quasigroup $(Q, \cdot)$ has the *weak commutative inverse property (WCIP)* with respect to a permutation J of Q if

$$J(x \cdot y) \cdot y = Jx \quad (4.10)$$

for all $x, y \in Q$ [20]. Note that (4.10) is true if and only if $((13), (J, \varepsilon, J))$ is an autostrophism of $(Q, \cdot)$. Therefore, WCIP-quasigroup is a ρ -inverse quasigroup with $\rho_2 = \varepsilon, \rho_1 = \rho_3 = J$ and $J^2 = \varepsilon$ (from Lemma 3.15 [20]). Hence in the WCIP-quasigroup, ${}^{(13)}N_l^A = J({}^{(13)}N_l^A J$ or ${}^{(13)}N_l^A \cong {}^{(13)}N_l^A, {}^{(23)}N_r^A = J({}^{(12)}N_m^A$, and ${}^{(12)}N_m^A = {}^{(23)}N_r^A J$.

A quasigroup $(Q, \cdot)$ is a μ -inverse quasigroup if there exist permutations μ_1, μ_2, μ_3 of the set Q such that

$$\mu_1 y \cdot \mu_2 x = \mu_3(x \cdot y) \quad (4.11)$$

for all $x, y \in Q$ [20]. It may be noted that (4.11) holds if and only if $((12), (\mu_1, \mu_2, \mu_3)) \in \text{Aus}(Q, \cdot)$.

From Theorem 3.3, we have in a μ -inverse quasigroup ${}^{(13)}N_l^A = \theta^{-1}({}^{(23)}N_r^A \theta, {}^{(23)}N_r^A = \theta^{-1}({}^{(13)}N_l^A \theta$ and ${}^{(12)}N_m^A = \theta^{-1}({}^{(12)}N_m^A \theta$, which implies left (13)-A-nucleus is isomorphic to right (23)-A-nucleus. Also observe that, $\theta^2 \in N({}^{(13)}N_l^A) \cap N({}^{(23)}N_r^A)$ and $\theta \in N({}^{(12)}N_m^A)$.

From Table 4, we conclude that in a μ -inverse quasigroup, we have ${}^{(13)}N_l^A = \mu_3^{-1}({}^{(23)}N_r^A \mu_1, {}^{(13)}N_l^A = \mu_1^{-1}({}^{(23)}N_r^A \mu_3, {}^{(23)}N_r^A = \mu_3^{-1}({}^{(13)}N_l^A \mu_2, {}^{(23)}N_r^A = \mu_2^{-1}({}^{(13)}N_l^A \mu_3, {}^{(12)}N_m^A = \mu_2^{-1}({}^{(12)}N_m^A \mu_1$, and ${}^{(12)}N_m^A = \mu_1^{-1}({}^{(12)}N_m^A \mu_2$.

5. Conclusions

In this paper, the characterization of the inverse sets of σ -A-nuclei and products of σ -A-nuclei and τ -A-nuclei of a quasigroup, and their respective component sets have been discussed (where $\sigma, \tau \in S_3$). Further, connections between the components of the σ -A-nuclei of a quasigroup and components of the σ -A-nuclei of its isostrophic images have been derived. Finally, we have investigated properties of σ -A-nuclei of various inverse quasigroups using the derived connections.

This study may further be carried out to obtain more relationships on σ -A-nuclei of a quasigroup. Also, the properties of the σ -A-nuclei may further be explored for various other quasigroup classes available in the literature.

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