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Perfect codes in poset spaces and poset block spaces

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ABSTRACT

The paper begins by giving a counter example to show that the algorithm for construction of new perfect poset codes from a given perfect poset code by removal of a coordinate as given by Lee (2004) [11] does not hold. The algorithm has been improved and generalized to obtain new perfect poset block codes from a given perfect poset block code. The modified necessary and sufficient conditions for the construction of new perfect poset codes have been derived as a particular case. A bound has been obtained on the height of poset P_s that turns a given π-code into r-perfect (P_s, π)-code. We show that there does not exist a poset which admits the binary Simplex code of order 3 to be a 2-perfect poset code. Also, all the poset structures which admit the extended ternary Golay code to be a 3-perfect poset code have been classified.

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1. Introduction

The theory of error-correcting codes deals with the study of linear subspaces of $\mathbb{F}_q^n$, where $\mathbb{F}_q^n$ denotes the space of all n-tuples over the finite field $\mathbb{F}_q$. Most of the studies in coding theory have been made on codes endowed with Hamming metric. An optimal class of codes having interesting mathematical structure is the class of perfect codes. A code is said to be perfect if balls of some particular radius centered at the codewords of C are disjoint and cover $\mathbb{F}_q^n$. Perfect codes find applications in group theory, combinatorial configurations and Diophantine number theory. One of the limitations of considering Hamming metric is the scarcity of perfect codes [14].

During 1990's, applications of codes in cryptography, experimental designs and high-dimensional numerical integration and more interestingly, the generalizations of the classical problems of coding theory motivated the researchers to explore codes endowed with metric other than Hamming metric. Using partially ordered sets and Niederreiter's generalization of the classical problem of coding theory [12], Brualdi et al. [2] introduced the concept of codes endowed with poset metric. The class of perfect codes with respect to poset metric is extremely large due to the increased packing radius in this case [5]. Moreover, MDS codes with respect to poset metric also occur in abundance [8]. In [7, 9,10], the approach has been to define poset on the set of n coordinates such that the given code in Hamming space turns out to be a perfect poset code. However, to the best of our knowledge, no general construction is known. In [11], Lee has given an algorithm to add or remove coordinates from a given perfect poset code resulting in new perfect codes. However, the result needs to be revised in view of the counter example given in section 3.

Block metric or π-metric, introduced by Feng et al. [6], is another generalization of Hamming metric. Encompassing poset and block structures, poset block metric was introduced by Alves et al. [1] and is by far the most generalized of all approaches, including the widely studied NRT block metric [13] which forms a subclass. It may be seen that the relationship between block metric and poset block metric is analogous to that of Hamming metric and poset metric. We explore the perfectness of block codes in poset block space.

In section 2, basic definitions have been presented. In section 3, we present a counter example to show that the algorithm given by Lee [11, Theorem 3.1(2)] does not hold. Generalizing the same, a modified algorithm to add or remove blocks from a given poset block metric so as to obtain new perfect codes has been derived. As a particular case, the corresponding necessary and sufficient conditions for the construction of new perfect poset codes have been obtained. In section 4, we present a bound on the height of poset which turns a given π-code into r-perfect poset block code. Using the above, we show that the binary Simplex code of order 3 can not be 2-perfect code with respect to poset metric. Also, all the poset structures which admit the extended ternary Golay code to be a 3-perfect poset code have been classified.

2. Preliminaries

In this section, basic definitions of poset block metric space have been provided.

Let $P_s = ([s], \leq)$ denote a **poset** with s elements, where $[s] := \{1, 2, \dots, s\}$ and $\leq$ is a partial order on $[s]$. An **ideal** I of the poset P_s is a subset of $[s]$ such that if $i \in I$ and $j \leq i$, then $j \in I$. For $X \subseteq [s]$, we denote by $\langle X \rangle$ the smallest ideal containing X, called the **ideal generated by** X. We prefer to denote the ideal generated by $X = \{i\}$ as $\langle i \rangle$ rather than $\langle \{i\} \rangle$ as in [1]. We denote by M_X the set of maximal elements of $X \subseteq P_s$. As per convention, we use the Hasse diagram to represent P_s graphically and call it P_s-graph. To give description of the poset P_s, we use the following notations:

$$\Gamma^{(t)}(P_s) = \{i \in [s] : |\langle i \rangle| = t\}$$

$$\Gamma_a(P_s) = \{i \in [s] : a < i\}$$

$$\Gamma_a^{(t)}(P_s) = \Gamma^{(t)}(P_s) \cap \Gamma_a(P_s).$$

For an integer ℓ with $1 \leq \ell \leq s$, we define the poset P_s^ℓ on $[s]$ as follows (Fig. 1):

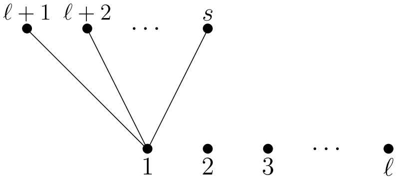

Fig. 1. P_s^ℓ-graph.

We have that:

$$\Gamma^{(1)}(P_s^\ell) = \{1, 2, \dots, \ell\}$$

$$\Gamma_1^{(2)}(P_s^\ell) = \{\ell + 1, \dots, s\}$$

$$\Gamma_a^{(2)}(P_s^\ell) = \phi \text{ if } 2 \leq a \leq \ell.$$

Let $\pi : [s] \to \mathbb{N}$ be a map such that $n = \sum_{i=1}^{s} \pi(i)$. The map π is said to be a **labeling** of the poset P_s, and the pair (P_s, π) is called a **poset block structure** over $[s]$. Let $\pi(i) = k_i$ and consider

$$V = \mathbb{F}_q^{k_1} \times \mathbb{F}_q^{k_2} \times \cdots \times \mathbb{F}_q^{k_s}$$

which is isomorphic to $\mathbb{F}_q^n$. Each $x \in \mathbb{F}_q^n$ can be written as $x = (x_1, x_2, \dots, x_s)$ where $x_i \in \mathbb{F}_q^{k_i}, 1 \leq i \leq s$. The **block support** or π**-support** and (P_s, π)**-weight** of $x \in \mathbb{F}_q^n$ are defined, respectively, as

$$\text{supp}_\pi(x) = \{i : 1 \leq i \leq s, x_i \neq 0\}$$

and

$$w_{(P_s,\pi)}(x) = |\langle \mathrm{supp}_\pi(x) \rangle|$$

where $|\cdot|$ denotes the cardinality of the given set. For $x, y \in \mathbb{F}_q^n$,

$$d_{(P_s,\pi)}(x,y) = w_{(P_s,\pi)}(x-y)$$

defines a metric over $\mathbb{F}_q^n$ called the **poset block metric** or (P_s,π)**-metric**. The pair $(\mathbb{F}_q^n, d_{(P_s,\pi)})$ is said to be a **poset block space**.

A k-dimensional subspace $C \subseteq \mathbb{F}_q^n$ is said to be an $(n,k,d_{(P_s,\pi)})$ (P_s,π)**-code**, where $\mathbb{F}_q^n$ is equipped with the poset block metric $d_{(P_s,\pi)}(\cdot,\cdot)$ and

$$d_{(P_s,\pi)}(C) = \min\{w_{(P_s,\pi)}(c) : 0 \neq c \in C\}.$$

When the label π satisfies $\pi(i) = 1$ for all $i \in [s]$, the poset block metric reduces to poset metric or P_n-metric introduced by Brualdi et al. [2]. Similarly, the block metric or π-metric introduced by Feng et al. [6] becomes a particular case of poset block metric when P_s is taken to be an anti-chain. When $\pi(i) = 1$ for all $i \in [s]$ and P_s is an anti-chain then the poset block metric reduces to the classical Hamming metric. We use subscript H to denote Hamming metric. Also, $\mathrm{supp}_H(x)$ denotes the set of all non-zero coordinates of $x \in \mathbb{F}_q^n$. When $|\langle i \rangle| = i$ for all $i \in [s]$, i.e., when the poset is a chain, the metric becomes NRT block metric [13]. In addition, if we take $\pi(i) = 1$ for all $i \in [s]$, the metric becomes NRT metric, a particular case of Rosenbloom–Tsfasman metric [15].

For a metric d on $\mathbb{F}_q^n$, the **ball** with center x and radius r is denoted by $B_d(x,r)$ and is defined as

$$B_d(x,r) = \{y \in \mathbb{F}_q^n : d(x,y) \leq r\}.$$

For all $x \in \mathbb{F}_q^n$,

$$|B_{P_n}(x,r)| = 1 + \sum_{i=1}^{r}\sum_{j=1}^{i}(q-1)^j q^{i-j}\Omega_j(i); \tag{2.1}$$

$$|B_\pi(x,r)| = 1 + \sum_{\lambda=1}^{r}\sum_{1\leq i_1\leq \ldots \leq i_\lambda \leq s}(q^{k_{i_1}}-1)\ldots(q^{k_{i_\lambda}}-1); \tag{2.2}$$

$$|B_{(P_s,\pi)}(x,r)| = 1 + \sum_{i=1}^{r}\sum_{j=1}^{i}\sum_{I\in\theta_j(i)}\prod_{m\in M_I}(q^{k_m}-1)\prod_{\ell<m;m\in M_I}q^{k_\ell}; \tag{2.3}$$

where $\Omega_j(i) = \{I \subseteq P_n : I \text{ ideal}, |I| = i, |M_I| = j\}$, $\theta_j(i) = \{I \subseteq P_s : I \text{ ideal}, |I| = i, |M_I| = j\}$.

The **packing radius** of a code $C \subseteq \mathbb{F}_q^n$ with respect to a metric d on $\mathbb{F}_q^n$ is defined as the largest non-negative integer $R_d(C)$ such that any two balls of radius $R_d(C)$ centered at distinct elements of C are disjoint. The **covering radius** of a code C with respect to a metric d is defined as the smallest non-negative integer ρ_d such that the balls of radius ρ_d centered at the codewords of C cover the whole space, i.e.,

$$\mathbb{F}_q^n = \bigcup_{c \in C} B_d(c, \rho_d).$$

A code is said to be an r**-perfect** code with respect to metric d if

$$\mathbb{F}_q^n = \bigsqcup_{c \in C} B_d(c, r)$$

i.e., the balls of radius r with respect to the metric d are disjoint (the partition condition) and cover the whole space (the sphere packing condition).

3. Algorithm for construction of new perfect codes

In this section, an algorithm for the construction of new perfect poset block codes from a given perfect poset block code has been derived. In particular, we modify the algorithm [11, Theorem 3.1] for the case of poset metric. Lee [11, Theorem 3.1] proved the following result. We denote by e_i the vector with 1 at i-th position and 0 elsewhere.

Result 3.1 *([11, Theorem 3.1]).*

(1) *Assume that C is an r-perfect (n, k) P_n-code. Consider the P'_{n+1}-poset constructed by connecting a new variable $n+1$ to some elements of the set $\{1, \dots, n\}$ in P_n-graph, and let $(n+1, k+1)$ P'_{n+1}-code $C' = (C, *)$ for $* \in \mathbb{F}_q$. Furthermore, we assume that $w_{P'_{n+1}}(e_i)$ (the weight of e_i given by poset P'_{n+1}) is the same as $w_{P_n}(e_i)$ (the weight of e_i given by poset P_n) for all $i = 1, \dots, n$. Then C' is an r-perfect P'_{n+1}-code if $w_{P'_{n+1}}(e_{n+1}) \geq r + 1$.*

(2) *Assume that C is an r-perfect (n, k) P_n-code and that the variable n is connected with some elements of the set $\{1, \dots, n\}$ in P_n-graph. Consider the P''_{n-1}-poset constructed by removing the variable n of P_n-graph, and let $(n-1, k-1)$ P''_{n-1}-code $C'' = \{(c_1, \dots, c_{n-1}) : (c_1, \dots, c_{n-1}, 0) \in C\}$. Furthermore we assume that $w_{P''_{n-1}}(e_i) = w_{P_n}(e_i)$ for all $i = 1, \dots, n-1$. Then C'' is an r-perfect P''_{n-1}-code or $C = \{(a_1 t, \dots, a_{n-1} t, t) : t \in \mathbb{F}_q\}$ for some fixed $a_i \in \mathbb{F}_q$. In particular if there is no perfect P''_{n-1}-code, then $C = \{(a_1 t, \dots, a_{n-1} t, t) : t \in \mathbb{F}_q\}$ for some fixed $a_i \in \mathbb{F}_q$.*

Firstly, we give a counter example to show that Result 3.1(2) [11, Theorem 3.1(2)] is not correct.

Example 3.1. Consider $\mathbb{F}_2^8$ with the poset metric determined by the following poset P_8 (Fig. 2):

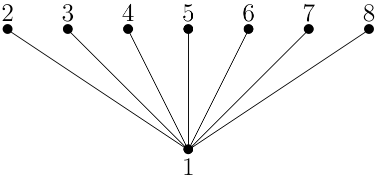

Fig. 2. P_8-graph.

Let $\mathcal{H}_3$ be the $(8,4,4)_H$ extended binary Hamming code with parity check matrix given by:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

Then as seen in [2, ex. p. 60], $\mathcal{H}_3$ is a 2-perfect P_8-code.

Let P_7'' be given by (Fig. 3):

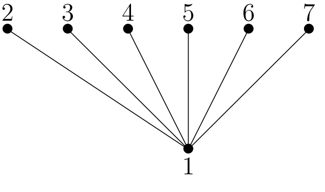

Fig. 3. P_7''-graph.

i.e., P_7'' is obtained by removing the 8-th element from P_8. Consider the code

$$\mathcal{H}_3'' = \{(c_1, \ldots, c_7) : (c_1, \ldots, c_7, 0) \in \mathcal{H}_3\}.$$

It can be seen that $\mathcal{H}_3''$ is a $(7,3,4)_H$ code. By Result 3.1(2) [11, Theorem 3.1(2)], we envisage that $\mathcal{H}_3''$ is a 2-perfect P_7''-code. But we have $d_{P_7''}((1,1,0,1,0,1,0), c) \geq 3$ for all $c \in \mathcal{H}_3''$. Therefore, $(1,1,0,1,0,1,0) \notin B_{P_7''}(c, 2)$ for any $c \in \mathcal{H}_3''$. This contradicts that $\mathcal{H}_3''$ is a 2-perfect P_7''-code. $\square$

We generalize [11, Theorem 3.1] to the case of poset block metric and develop an algorithm to construct new perfect codes by adding (or removing) a block to (or from) the given perfect poset block code. The correction to Result 3.1(2) [11, Theorem 3.1(2)] has been provided as a particular case of the following theorem. The following lemma is easy to observe.

Lemma 3.1. *Consider $\mathbb{F}_q^n$ endowed with (P_s, π)-metric.*

- **(i)** *Consider the P'_{s+1}-poset constructed by connecting a new variable $s+1$ to some elements of the set $\{1, \ldots, s\}$ in P_s-graph. Also, assume π' to be a label satisfying $\pi'(i) = \pi(i)$ for all $i \in [s]$ and $\pi'(s+1) = k_{s+1}$. Then the (P'_{s+1}, π')-metric on $\mathbb{F}_q^n \times \mathbb{F}_q^{k_{s+1}}$ satisfies $w_{(P'_{s+1},\pi')}(x) = w_{(P_s,\pi)}(x)$ for all $x \in \mathbb{F}_q^n$ if and only if $s+1 \in M_{P'_{s+1}}$.*
- **(ii)** *Consider the P''_{s-1}-poset constructed by removing the variable i of P_s-graph where $i \in [s]$. Also, assume π'' to be a label satisfying $\pi''(j) = \pi(j)$ for all $j \in P''_{s-1}$. Then the (P''_{s-1}, π'')-metric on $\mathbb{F}_q^{n-k_i}$ satisfies $w_{(P''_{s-1},\pi'')}(x) = w_{(P_s,\pi)}(x)$ for all $x \in \mathbb{F}_q^{n-k_i}$ if and only if $i \in M_{P_s}$.*

Theorem 3.1. *Assume that $C \subseteq \mathbb{F}_q^n$ is an r-perfect (n, k) (P_s, π)-code.*

- **(i)** *Let P'_{s+1} be a poset constructed by connecting a new variable $s+1$ to some elements of the set $\{1, \ldots, s\}$ in P_s-graph such that $s+1 \in M_{P'_{s+1}}$. Also, assume π' to be a label satisfying $\pi'(i) = \pi(i)$ for all $i \in [s]$ and $\pi'(s+1) = k_{s+1}$. Consider the $(n + k_{s+1}, k + k_{s+1})$ code $C' = \{(c, *)$ for $c \in C$ and $* \in \mathbb{F}_q^{k_{s+1}}\}$. Then C' is an r-perfect (P'_{s+1}, π')-code if and only if $|\langle s+1\rangle| \geq r+1$.*
- **(ii)** *Let P''_{s-1} be a poset constructed by removing the variable i of P_s-graph such that $i \in M_{P_s}$. Also, assume π'' to be a label satisfying $\pi''(j) = \pi(j)$ for all $j \in P''_{s-1}$. Assume that $k > k_i$ and there does not exist a coordinate position which is zero for all codewords of C. Consider the $(n - k_i, k - k_i)$ (P''_{s-1}, π'')-code*

$$C'' = \{(c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_s) : (c_1, \ldots, c_{i-1}, 0, c_{i+1}, \ldots, c_s) \in C\}.$$

  Then C'' is an r-perfect (P''_{s-1}, π'')-code if and only if $|\langle i\rangle| \geq r+1$.

Proof. **(i)** Since $s+1 \in M_{P'_{s+1}}$, using Lemma 3.1(i) we have $w_{(P'_{s+1},\pi')}(x) = w_{(P_s,\pi)}(x)$ for all $x \in \mathbb{F}_q^n$. Firstly, assume that $|\langle s+1\rangle| \geq r+1$. If $x \in \mathbb{F}_q^{n+k_{s+1}}$ such that $s+1 \in \operatorname{supp}_\pi(x)$, then $w_{(P'_{s+1},\pi')}(x) \geq r+1$. Hence

$$|B_{(P'_{s+1},\pi')}(0, r)| = |B_{(P_s,\pi)}(0, r)| = q^{n-k}.$$

This implies that the balls of radius r with respect to (P'_{s+1}, π') metric cover $\mathbb{F}_q^{n+k_{s+1}}$. We need to show the partition condition. Let, if possible,

$$(x_1, x_2, \ldots, x_{s+1}) \in B_{(P'_{s+1},\pi')}((c_1, c_2, \ldots, c_{s+1}), r) \cap B_{(P'_{s+1},\pi')}((\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_{s+1}), r).$$

Then $w_{(P'_{s+1},\pi')}(x_1 - c_1, x_2 - c_2, \ldots, x_{s+1} - c_{s+1}) \leq r$ and $w_{(P'_{s+1},\pi')}(x_1 - \hat{c}_1, x_2 - \hat{c}_2, \ldots, x_{s+1} - \hat{c}_{s+1}) \leq r$. Since $|\langle s+1\rangle| \geq r+1$, therefore $x_{s+1} = c_{s+1} = \hat{c}_{s+1}$. Thus we have

$$(x_1, x_2, \ldots, x_s) \in B_{(P_s,\pi)}((c_1, c_1, \ldots, c_s), r) \cap B_{(P_s,\pi)}((\hat{c}_1, \hat{c}_1, \ldots, \hat{c}_s), r)$$

which is a contradiction to the fact that C is an r-perfect (P_s, π)-code. Hence, the partition condition holds and C' is an r-perfect (P'_{s+1}, π')-code.

Conversely, let C' be an r-perfect (P'_{s+1}, π')-code. Let, if possible, $|\langle s+1\rangle| < r+1$. Take any non-zero vector $v \in \mathbb{F}_q^{k_{s+1}}$. Then $v \in C'$ and

$$v \in B_{(P'_{s+1},\pi')}(0, r) \cap B_{(P'_{s+1},\pi')}(v, r).$$

This contradicts that C' is an r-perfect (P'_{s+1}, π')-code. Therefore, $|\langle s+1\rangle| \geq r+1$.

(ii) Since $i \in M_{P_s}$, using Lemma 3.1(ii) we have $w_{(P''_{s-1},\pi'')}(x) = w_{(P_s,\pi)}(x)$ for all $x \in \mathbb{F}_q^{n-k_i}$. Firstly, assume that $|\langle i\rangle| \geq r+1$. Let $x \in \mathbb{F}_q^{n-k_i} \subseteq \mathbb{F}_q^n$. Since C is an r-perfect (P_s, π)-code there exists a unique $c \in C$ such that $x \in B_{(P_s,\pi)}(c, r)$. Since $|\langle i\rangle| \geq r+1$, we must have $c_i = 0$. Thus $(c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_s) \in C''$ and $x \in B_{(P''_{s-1},\pi'')}((c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_s), r)$. This implies that

$$\mathbb{F}_q^{n-k_i} = \bigsqcup_{c'' \in C''} B_{(P''_{s-1},\pi'')}(c'', r).$$

Conversely, let C'' be an r-perfect (P''_{s-1}, π'')-code. Since both C and C'' are r-perfect, we have $|B_{(P_s,\pi)}(0, r)| = |B_{(P''_{s-1},\pi'')}(0, r)| = q^{n-k}$. Also, $B_{(P''_{s-1},\pi'')}(0, r) \subseteq B_{(P_s,\pi)}(0, r)$. This implies that $B_{(P''_{s-1},\pi'')}(0, r) = B_{(P_s,\pi)}(0, r)$. Let, if possible, $|\langle i\rangle| < r+1$. Take any non-zero vector $u \in \mathbb{F}_q^{k_i}$. Then $u \in B_{(P_s,\pi)}(0, r)$ whereas $u \notin B_{(P''_{s-1},\pi)}(0, r)$, which is a contradiction. Thus $|\langle i\rangle| \geq r+1$. □

As a particular case of the above result, we now provide the modified version of Result 3.1 [11, Theorem 3.1].

Theorem 3.2. *Assume that $C \subseteq \mathbb{F}_q^n$ is an r-perfect (n, k) P_n-code.*

- **(i)** *Let P'_{n+1} be a poset constructed by connecting a new variable $n+1$ to some elements of the set $\{1, \ldots, n\}$ in P_n-graph such that $n+1 \in M_{P'_{n+1}}$. Consider the $(n+1, k+1)$ code $C' = \{(c, *)$ for $c \in C$ and $* \in \mathbb{F}_q\}$. Then C' is an r-perfect P'_{n+1}-code if and only if $|\langle n+1\rangle| \geq r+1$.*
- **(ii)** *Let P''_{n-1} be a poset constructed by removing the variable i of P_n-graph such that $i \in M_{P_n}$. Assume that $k > 1$ and there does not exist a coordinate position which is zero for all codewords of C. Consider the $(n-1, k-1)$ code*

$$C'' = \{(c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n) : (c_1, \ldots, c_{i-1}, 0, c_{i+1}, \ldots, c_n) \in C\}.$$

Then C'' is an r-perfect P''_{n-1}-code if and only if $|\langle i\rangle| \geq r+1$.

Remark 3.1. It is worth noting that the converse of Result 3.1 [11, Theorem 3.1] is also true.

Remark 3.2. When $|\langle i \rangle| = 1$ for all $i \in [s]$, i.e., for the case of block metric, it is clear from Theorem 3.1 that no new perfect codes can be obtained by removing a block from a given r-perfect code. However, new perfect codes can be obtained by adding a block $s+1$ with $|\langle s+1 \rangle| = 2$, which shall result in a poset block metric. Further, when $\pi(i) = 1$ for all $i \in [s]$, i.e., for the case of Hamming metric, it is clear from Theorem 3.2 that no new r-perfect codes can be constructed from a given r-perfect code by removing a coordinate. However, new perfect codes can be obtained by adding a coordinate $n + 1$ with $|\langle n + 1 \rangle| = 2$, which shall result in a non-Hamming metric as illustrated in the following example.

Example 3.2. Consider $\mathbb{F}_2^7$ with the poset metric determined by the anti-chain poset P_7 which is nothing but Hamming metric. Let C be the $(7, 4, 3)_H$ Hamming code generated by the following generator matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

It is a 1-perfect code with respect to Hamming metric. Consider the following poset P'_8 (Fig. 4) formed by adding a coordinate to P_7:

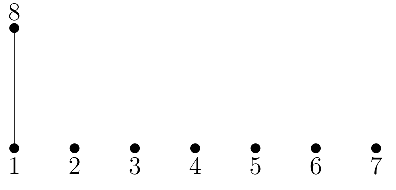

Fig. 4. P'_8-graph.

Since $|\langle 8 \rangle| = 2$, using Theorem 3.2 we have that the code $C' = \{(c, *)$ for $c \in C$ and $* \in \mathbb{F}_2\}$ is a 1-perfect P'_8-code. $\square$

Remark 3.3. Let P_s be the chain poset and C be an r-perfect NRT block code such that no coordinate in C is all zero. Since $M_{P_s} = \{s\}$ and $|\langle s \rangle| > r$, it follows from Theorem 3.1 that only the s-th block can be removed provided $k > k_s$. Blocks are added by extending the chain, i.e., $(s+1)$-th block can be added at $(s+1)$-th level of the poset P'_{s+1}. Further, when $\pi(i) = 1$ for all $i \in [s]$, the metric reduces to NRT metric. Let P_n be the chain poset and C be an r-perfect NRT code such that no coordinate in C is all zero. Using [2, Theorem 2.1], we get $r = n - k$. Since $M_{P_n} = \{n\}$ and $|\langle n \rangle| > n - k$, it follows from Theorem 3.2 that only the n-th coordinate can be removed provided $k > 1$. Similarly, all the coordinates up to $n - k$ can be removed one by one. Coordinates can be added by extending the chain, i.e., $(n+1)$-th coordinate can be added at $(n+1)$-th level of the poset P'.

We now present an example to illustrate Theorem 3.1.

Example 3.3. Consider $\mathbb{F}_3^4$ with the (P_3, π) metric where P_3 is given by (Fig. 5):

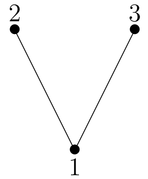

Fig. 5. P_3-graph.

with $\pi(1) = 2$ and $\pi(2) = \pi(3) = 1$. Let C be the $(4, 2)$ code generated by the following generator matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

It is clear that $d_{(P_3,\pi)}(C) = 2$. Using the formula for the packing radius of a hierarchical poset block code given in [4, Theorem 3.9], we get $R_{(P_3,\pi)}(C) = 1$. Using (2.3), we have

$$\begin{aligned} |B_{(P_3,\pi)}(0, 1)| &= 1 + (3^2 - 1) \\ &= 3^2. \end{aligned}$$

Thus, the balls of radius 1 are disjoint and cover $\mathbb{F}_3^4$. Hence, C is a 1-perfect (P_3, π)-code. Now $M_{P_3} = \{2, 3\}$. Let $P_2'' = \{1, 2\}$. By Theorem 3.1(i), we have that the $(3, 1)$ code $C'' = \{(00, 0), (11, 2), (22, 1)\}$ is a 1-perfect (P_2'', π'')-code. Similarly, removing the second block yields a 1-perfect code, namely, $C'' = \{(00, 0), (21, 2), (12, 1)\}$.

Addition of another block of any dimension can be done in the following ways (Fig. 6):

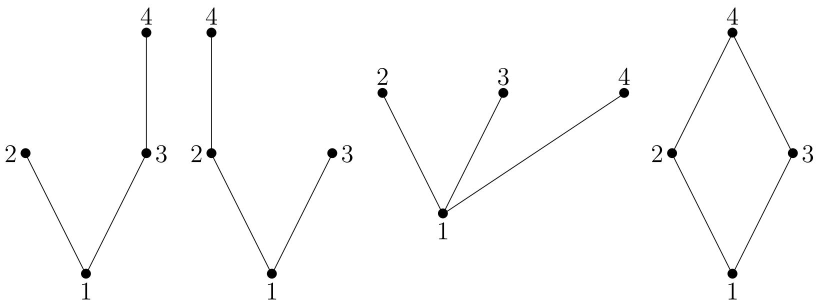

Fig. 6. P_4-graphs.

In each case, using Theorem 3.1(ii) we have that the $(5, 3)$ code $C' = \{(c, *) : c \in C, * \in \mathbb{F}_3^{k_4}\}$ is a 1-perfect (P_4', π')-code. $\square$

As a corollary to Result 3.1 [11, Theorem 3.1], Lee proved the following result.

Result 3.2 *([11, Corollary 3.2]). Let P_n be the poset with n-elements, $1 < 2 < \cdots < n$. Then C is a perfect P_n-code if and only if C is the type of $\{(0, \ldots, 0, t_{n-k+1}, \ldots, t_n) : t_i \in \mathbb{F}_q\}$.*

We provide a counter example to show that the above result does not hold.

Example 3.4. Consider $\mathbb{F}_2^5$ with P_5-metric where P_5 is the chain poset. Let $C \subseteq \mathbb{F}_2^5$ be the code generated by the following generator matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then C is a $(5, 3)$ code with $d_{P_5}(C) = 3 = n - k + 1$. Using [13, Theorem 5], $R_{P_5}(C) = 2$. Also, using (2.1) we get

$$\begin{aligned} |B_{P_5}(0, 2)| &= 1 + \Omega_1(1) + 2\,\Omega_1(2) \\ &= 4 \end{aligned}$$

which implies that

$$|C|.|B_{P_5}(0, 2)| = 2^5.$$

Hence, C is a 2-perfect P_5-code although C is not one of the codes mentioned in [11, Corollary 3.2]. $\square$

Remark 3.4. The correction to [11, Corollary 3.2] is [2, Theorem 3.1].

4. Perfectness of a block code in poset block space

It is well known that the extended binary Hamming code, $\mathcal{H}_m$, and the extended binary Golay code, $\mathcal{G}_{24}$, are not perfect with respect to Hamming metric. In [7,9,10], the authors have investigated whether there exists a poset on the set of coordinates which admit these code to be r-perfect poset codes for different values of r. We note that the relationship between block metric and poset block metric is analogous to that of Hamming metric and poset metric in the sense that the poset has to be defined on the set of blocks instead of coordinates. Using the approach used for $\mathcal{H}_m$ and $\mathcal{G}_{24}$, we now investigate whether there exists a poset on the set of blocks of a π-code C which turns the code into a perfect poset block code.

Proposition 4.1. *Let C be a code which is not perfect with respect to π-metric. Then there does not exist a poset on the set of blocks of π which admit C to be an r-perfect poset block code for $r \leq \left\lfloor \frac{d_\pi(C)-1}{2} \right\rfloor$.*

Proof. Since C is not perfect with respect to π-metric, we have

$$\rho_\pi(C) > R_\pi(C). \tag{4.1}$$

Let P_s be a poset defined on the set of blocks of π. From [4, Proposition 3.10], we have

$$B_{(P_s,\pi)}(x,r) \subseteq B_\pi(x,r).$$

Therefore, the covering radius of a code C with respect to (P_s,π)-metric is always greater than or equal to that with respect to π-metric, i.e.,

$$\rho_{(P_s,\pi)}(C) \geq \rho_\pi(C). \tag{4.2}$$

From (4.1) and (4.2), we get

$$\rho_{(P_s,\pi)}(C) > R_\pi(C).$$

Thus, the code can not be r-perfect (P_s,π)-code for $r \leq \left\lfloor \frac{d_\pi(C)-1}{2} \right\rfloor$ since $R_\pi(C) = \left\lfloor \frac{d_\pi(C)-1}{2} \right\rfloor$ as seen in [3]. $\square$

In the following proposition, we generalize [7, Lemma 1] and [9, Lemma 1] to obtain a bound on the height of poset which turns a given π-code into an r-perfect (P_s,π)-code.

Proposition 4.2. *Let C be an (n,k) π-code. Let r be a positive integer such that $\left\lfloor \frac{d_\pi(C)-1}{2} \right\rfloor < r < d_\pi(C) - 1$. If a poset P_s defined on the set of blocks of π turns C into an r-perfect (P_s,π)-code then*

$$P_s = \Gamma^{(1)}(P_s) \cup \ldots \cup \Gamma^{(r-t)}(P_s)$$

where $t = d_\pi(C) - (r+2)$.

Proof. Since $\left\lfloor \frac{d_\pi(C)-1}{2} \right\rfloor < r < d_\pi(C) - 1$, we have $t \geq 0$ and $r - t > 0$. We prove the result by contradiction. Consider the following two cases:
Case I When $t = 0$.
Assume that there exists a block i with $|\langle i \rangle| > r$. Let $x \in \mathbb{F}_q^{k_i}$ be any non-zero vector. Then $w_\pi(x) = 1$ and $w_{(P_s,\pi)}(x) \geq r+1$. Since $d_\pi(C) = r+2$, therefore $x \notin C$. Since C is an r-perfect (P_s,π)-code, there exists $c \in C$ such that $x \in B_{(P_s,\pi)}(c,r)$. Also $w_{(P_s,\pi)}(x) \geq r+1$ implies that $c \neq 0$. Therefore, $w_\pi(c) \geq r+2$. We get

$$\begin{aligned} d_{(P_s,\pi)}(x,c) &\geq d_\pi(x,c) \\ &= w_\pi(x-c) \\ &\geq r+1 \end{aligned}$$

which is a contradiction.

Case II When $t \neq 0$.
Suppose that there exists a block i such that $|\langle i\rangle| > r - t$.

If $|\langle i\rangle| \geq r + 1$, take t blocks $j_1, j_2, \dots, j_t \in [s] \setminus \{i\}$.
If $|\langle i\rangle| < r + 1$, take t blocks $j_1, j_2, \dots, j_t \in [s] \setminus \langle i\rangle$.

Take x such that $\mathrm{supp}_\pi(x) = \{i, j_1, j_2, \dots, j_t\}$. Then $w_\pi(x) = t+1$ and $w_{(P_s,\pi)}(x) \geq r+1$. Since $d_\pi(C) = r + t + 2$, therefore $x \notin C$. Since C is an r-perfect (P_s, π)-code, there exists $c \in C$ such that $x \in B_{(P_s,\pi)}(c, r)$. Repeating the argument as in *Case I*, we get the result. □

Remark 4.1. It is worth noting that for $d_\pi(C) \leq 3$, there does not exist any $r \in \mathbb{Z}^+$ such that $\left\lfloor \frac{d_\pi - 1}{2} \right\rfloor < r < d_\pi - 1$.

Corollary 4.1. *For $r = \left\lfloor \frac{d_\pi(C)-1}{2} \right\rfloor + 1$ (i.e., $r = R_\pi(C)+1$), the above proposition reduces to*

$$P_s = \begin{cases} \Gamma^{(1)}(P_s) \cup \Gamma^{(2)}(P_s), & \text{if } d_\pi(C) \text{ is even} \\ \Gamma^{(1)}(P_s) \cup \Gamma^{(2)}(P_s) \cup \Gamma^{(3)}(P_s), & \text{if } d_\pi(C) \text{ is odd.} \end{cases}$$

Using Corollary 4.1 and the parameters of a given code, we may ascertain if there exists any poset such that a given π-code turns into an r-perfect poset block code for $r = R_\pi(C) + 1$ as illustrated in the following example.

Example 4.1. Consider $\mathbb{F}_2^{25}$ with π-metric where π consists of 5 blocks each of dimension 5, i.e., $\pi(1) = \pi(2) = \pi(3) = \pi(4) = \pi(5) = 5$. Consider the code C generated by the generator matrix with five identical blocks given by:

$$G = [G_1, G_2, G_3, G_4, G_5]$$

where

$$G_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, 1 \leq i \leq 5.$$

It is clear that C is a $(25, 5)$ π-code with $d_\pi(C) = 5$. Then $R_\pi(C) = 2$. Using (2.2), we get

$$\begin{aligned} |B_\pi(0, 2)| &= 1 + \sum_{i=1}^{2} \binom{5}{i} (2^5 - 1)^i \\ &= 9766. \end{aligned}$$

Since $2^{20} > |B_\pi(0, 2)|$, therefore C is not a 2-perfect π-code.

Let P_5 be a poset defined on $[5]$ which turns C into a 3-perfect (P_5, π)-code. By Corollary 4.1, we have

$$P_5 = \Gamma^{(1)}(P_5) \cup \Gamma^{(2)}(P_5) \cup \Gamma^{(3)}(P_5).$$

Let $i \in \Gamma^{(3)}(P_5)$. Take $j \in [5] \setminus \langle i \rangle$. Consider $v \in \mathbb{F}_2^{25}$ such that $i, j \in \mathrm{supp}_\pi(v)$ with $v_i \neq v_j$. Then $d_{(P_5,\pi)}(0, v) \geq 4$. Since $d_\pi(C) = 5$, $d_{(P_5,\pi)}(c, v) \geq 4$ for any $c \neq 0$, $c \in C$. Therefore, $v \notin B_{(P_5,\pi)}(c, 3)$ for any $c \in C$, which is a contradiction. Hence, $\Gamma^{(3)}(P_5) = \phi$.

Let $i \in \Gamma_a(P_5)$ and $j \in \Gamma_b(P_5)$ for any two distinct elements a, b in $\Gamma^{(1)}(P_5)$. Take $v \in \mathbb{F}_2^{25}$ such that $i, j \in \mathrm{supp}_\pi(v)$ with $v_i \neq v_j$. As shown earlier, $v \notin B_{(P_5,\pi)}(c, 3)$ for any $c \in C$, which is a contradiction. Therefore, we have that P_5 is equivalent to P_5^ℓ for some $1 \leq \ell \leq 4$. It may be noted that $\ell \neq 5$ else the metric reduces to block metric. Using the sphere packing condition, we get

$$|B_{(P_5,\pi)}(0, 3)| = 2^{20}.$$

Using (2.3), we get

$$1 + \sum_{i=1}^{3} \sum_{j=1}^{i} 31^j 32^{i-j} \theta_j(i) = 1048576.$$

It follows that

$$\begin{aligned} &1 + 31\binom{\ell}{1} + 31.32\binom{5-\ell}{1} + 31^2\binom{\ell}{2} + 0 \\ &\quad + 31^2.32\left[\binom{5-\ell}{1}\binom{\ell-1}{1} + \binom{5-\ell}{2}\right] + 31^3\binom{\ell}{3} = 1048576 \end{aligned}$$

which reduces to

$$29791\ell^3 - 178746\ell^2 + 327701\ell - 5339130 = 0.$$

The above equation has no integral solution as confirmed through Mathematica Software. Hence, there does not exist any poset which turns the given block code into a 3-perfect (P_5, π)-code. $\square$

Although Corollary 4.1 holds for a π-code, it is usually applied to check the perfectness in poset space of a code whose Hamming distance is known. In what follows, we discuss the existence of poset structures for binary Simplex code and extended ternary Golay code to be perfect. We denote by $S_m(2)$ the binary Simplex code of order m having parameters $(2^m - 1, m, 2^{m-1})_H$.

Theorem 4.1. *There are no poset structures which admit the binary Simplex code $S_3(2)$ to be a 2-perfect poset code.*

Proof. In general, for the binary Simplex code of order m, we have $R_H(S_m(2)) = 2^{m-2} - 1$. Therefore, we take $r = 2^{m-2}$. Let P_{2^m-1} be a poset that turns the binary Simplex code $S_m(2)$ into a 2^{m-2}-perfect poset code. Using Corollary 4.1, we have

$$P_{2^m-1} = \Gamma^{(1)}(P_{2^m-1}) \cup \Gamma^{(2)}(P_{2^m-1}).$$

Let $|\Gamma^{(1)}(P_{2^m-1})| = t$. Then $|\Gamma^{(2)}(P_{2^m-1})| = 2^m - 1 - t$. Using the sphere packing condition, we have

$$|B_{P_{2^m-1}}(0, 2^{m-2})| = 2^{2^m-1-m} \tag{4.3}$$

In order to prove this theorem, in particular for $S_3(2)$, (4.3) reduces to

$$|B_{P_7}(0, 2)| = 2^4.$$

Using (2.1), we get

$$1 + \sum_{i=1}^{2} \sum_{j=1}^{i} 2^{i-j} \Omega_j(i) = 16.$$

It follows that

$$1 + \binom{t}{1} + 2\binom{7-t}{1} + \binom{t}{2} = 16$$

which reduces to

$$t^2 - 3t - 2 = 0.$$

The above equation has no integral roots. Hence, there does not exist a poset which admits $S_3(2)$ to be a 2-perfect poset code. □

Now we investigate the poset structures that turn the extended ternary Golay code $\mathcal{G}_{12}$ with parameters $(12, 6, 6)_H$ into a 3-perfect poset code since $R_H(\mathcal{G}_{12}) = 2$.

Theorem 4.2. *The extended ternary Golay code $\mathcal{G}_{12}$ is a 3-perfect P_{12}-code if and only if P_{12} is equivalent to P_{12}^{ℓ} for some ℓ, $1 \leq \ell \leq 3$.*

To prove the theorem, we require the following lemmas.

Lemma 4.1. *If a poset P_{12} turns the extended ternary Golay code $\mathcal{G}_{12}$ into a 3-perfect P_{12}-code, then $P_{12} = \Gamma^{(1)}(P_{12}) \cup \Gamma^{(2)}(P_{12})$.*

Proof. The proof follows from Corollary 4.1. □

Lemma 4.2. *If the extended ternary Golay code $\mathcal{G}_{12}$ is a 3-perfect P_{12}-code, then for any two distinct elements a, b in $\Gamma^{(1)}(P_{12})$, we have either $\Gamma_a(P_{12}) = \phi$ or $\Gamma_b(P_{12}) = \phi$.*

Proof. Let, if possible, $\Gamma_a(P_{12}) \neq \phi$ and $\Gamma_b(P_{12}) \neq \phi$ for any two distinct elements a, b in $\Gamma^{(1)}(P_{12})$. Take $i \in \Gamma_a(P_{12})$ and $j \in \Gamma_b(P_{12})$. Take $v \in \mathbb{F}_3^{12}$ such that $i, j \in \mathrm{supp}_H(v)$. Then $d_{P_{12}}(0, v) = 4$. Since $d_H(\mathcal{G}_{12}) = 6$, $d_{P_{12}}(c, v) \geq 4$ for any $c \neq 0$, $c \in \mathcal{G}_{12}$. Therefore, $v \notin B_{P_{12}}(c, 3)$ for any $c \in \mathcal{G}_{12}$, which is a contradiction. □

Proof of the Theorem. Let $\mathcal{G}_{12}$ be a 3-perfect P_{12}-code. By Lemma 4.1 and 4.2, P_{12} is equivalent to P_{12}^{ℓ} for some ℓ, $1 \leq \ell \leq 11$. It may be noted that $\ell \neq 12$ else the metric reduces to Hamming metric. Using the sphere packing condition, we get

$$|B_{P_{12}^{\ell}}(0, 3)| = 3^6.$$

Using (2.1), we get

$$1 + \sum_{i=1}^{3} \sum_{j=1}^{i} 2^j 3^{i-j} \Omega_j(i) = 729.$$

It follows that

$$\begin{aligned} 1 + 2\binom{\ell}{1} + 2.3\binom{12-\ell}{1} + 2^2\binom{\ell}{2} + 0 \\ + 2^2.3\left[\binom{12-\ell}{1}\binom{\ell-1}{1} + \binom{12-\ell}{2}\right] + 2^3\binom{\ell}{3} = 729 \end{aligned}$$

which reduces to

$$(\ell - 1)(\ell - 2)(\ell - 3) = 0.$$

Thus, we have that $\ell = 1, 2, 3$ are the only permissible values.

Conversely, let P_{12} be equivalent to P_{12}^{ℓ} for some ℓ, $1 \leq \ell \leq 3$. Then it is clear that the balls of radius 3 with respect to P_{12}^{ℓ}-metric cover $\mathbb{F}_3^{12}$. We show that the balls are disjoint. Let, if possible,

$$B_{P_{12}^{\ell}}(0, 3) \cap B_{P_{12}^{\ell}}(c, 3) \neq \phi$$

for some non-zero $c \in \mathcal{G}_{12}$. Take $x \in B_{P_{12}^{\ell}}(0, 3) \cap B_{P_{12}^{\ell}}(c, 3)$. Then $w_{P_{12}^{\ell}}(x) \leq 3$ and $w_{P_{12}^{\ell}}(c-x) \leq 3$. Since $w_H(c) \geq 6$ and $w_H(x) \leq w_{P_{12}^{\ell}}(x) \leq 3$, we have $w_{P_{12}^{\ell}}(c-x) \leq 3$ only if $w_H(c) = 6$, $w_H(x) = 3$ and $\mathrm{supp}_H(x) \subset \mathrm{supp}_H(c)$ which implies that $w_H(c - x) = 3$. Since $w_H(x) = 3$ and $w_{P_{12}^{\ell}}(x) \leq 3$, $1 \in \mathrm{supp}_H(x)$. Since $w_H(c - x) = 3$ and $1 \notin \mathrm{supp}_H(c - x)$, we get $\mathrm{supp}_H(c - x) \cap \Gamma^{(2)}(P_{12}^{\ell}) \neq \phi$. Therefore, $w_{P_{12}^{\ell}}(c - x) \geq 4$, which is a contradiction. Thus the balls of radius 3 with respect to P_{12}^{ℓ}-metric are disjoint and cover $\mathbb{F}_3^{12}$. Hence, $\mathcal{G}_{12}$ is a 3-perfect P_{12}^{ℓ}-code. □

5. Conclusion

The paper revolves around perfectness of codes in poset block space wherein either new perfect codes have been constructed from a given perfect code or a poset has been introduced to turn a code into perfect code. In the process, an analogous result on perfect codes in poset space has been revised.

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